Applications of h-principle to Symplectic and Contact geometry

Venkata Sai Narayana

December 19, 2021

Abstract

This is an article which will consist of typed up notes of lectures on Symplectic and Contact geometry.

1 Introduction

Definition 1 *A symplectic vector bundle is a tuple* $(V, M, \pi : V \to M, w)$ *where*

1. $\pi : V \to M$ *is a smooth vector bundle.*
2. $\omega : V \times V \to \mathbb{R}$ *is a smooth non degenerate skew symmetric* $2-$*form.* ω *smooth means for locally defined sections* v_1, v_2 *of* V*, the map*

$$x \to \omega_x(v_{1,x}, v_{2,x})$$

is smooth.

Now we give a nice characterization of the fact that ω is non degenerate.

Lemma 2 ω_x *is non degenerate if and only if* $\omega_x^n \neq 0$.

Proof: Suppose ω_x is non degenerate then we can get a symplectic basis $e_1, ..., e_n, f_1, ..., f_n$. The way to do this is simple and clearly explained in the Notes on Symplectic Geometry. Now once we have this it is obvious that $\omega_x^n(e_1, ..., e_n, f_1, ..., f_n) \neq 0$ since the form $\omega_x = \sum e_i \wedge f_i$ and therefore $\omega_x^n = e_1 \wedge ... \wedge e_n \wedge f_1 \wedge ... \wedge f_n$. Now it is clear that the value is just the determinant of the bilnear form in this basis which is 1.

Now we prove the converse. Suppose $\omega_x^n \neq 0$. If ω is degenerate, then there is a v such that $\omega_x(v, w) = 0$ for all w. Now complete v to a basis. Then we claim that $\omega_x^n(v, v_1, ..., v_{2n-1}) = 0$. This would be products of all possible two tuples and one of the terms in the product will contain v. So the product is zero. ∎

Lemma 3 *Every complex vector bundle can be given the structure of a symplectic vector bundle. The converse is also true but much harder to prove.*

Proof: Firstly here we will think of a complex vector bundle as real vector bundle with a map J inducing the complex structure. Now we will put a hermitian metric on this vector bundle that is a g such that $g(v, w) = g(Jv, Jw)$. This can be done by first putting some metric and then define $h(v, w) = \frac{g(v,w)+g(Jv,Jw)}{2}$. Now define a symplectic form to be $\omega(v, w) = g(v, Jw)$. ∎

Remark 4 *This theorem implies that the classifying space of both symplectic and complex vector bundles is $BU(n)$.*

Definition 5 *By a contact manifold $(M, \alpha = 0)$ we mean M together with a codimension 1 distribution $\xi = Ker(\alpha)$ which satisfies the property that $(\xi, d\alpha)$ is a symplectic vector bundle.*

Remark 6 *Here we are only considering the case of a coorientable contact structures that is those which are given by vanishing of a globally defined $1-$form.*

Lemma 7 *$(M, \alpha = 0)$ is contact if and only if $\alpha \wedge (d\alpha)^n \neq 0$.*

Proof: Suppose $(M, \alpha = 0)$ is contact. Then we know that $(d\alpha)^n$ is non degenerate. Now take a basis of $\alpha = 0$ say $w_1, ...w_{2n}$ and extending it to a basis by adding v such that $\alpha(v) \neq 0$. Now it is clear that $\alpha \wedge (d\alpha)^n(v, w_1, ..., w_{2n})$ is non zero since the terms which contain $d\alpha$ acting on v will also contain $\alpha(w_i)$ which is clearly zero. So the only non zero term left is $\alpha(v)d\alpha(w_1, .., w_{2n})$ which is clearly non zero.

Now we will prove the converse. Suppose $\alpha \wedge (d\alpha)^n \neq 0$, then we need to show that $d\alpha$ is a symplectic form on $\alpha = 0$. Now for that we just have to verify it is non degenerate. So we need to check that $d\alpha^n \neq 0$ on $\alpha = 0$. Suppose it is, then chose a basis $w_1, ..., w_{2n}$ and v as before. Now $\alpha \wedge (d\alpha)^n(v, w_1, ..., w_{2n}) = 0$ which is a contradiction. So $d\alpha$ is non degenerate. ■

Remark 8 *Contact structures are non integrable!! In fact they are as fas as they can be from being integrable. To prove that they are non integrable we can use the frobenius theorem. Suppose the distribution is involutive, then $v, w \in \alpha = 0$ implies $\alpha([v, w]) = 0$. Now this implies $d\alpha(v, w) = 0$ for all $v, w \in \alpha = 0$. This implies it is non degenerate.*

Remark 9 *What I meant when I said it is as far as it can be from being integrable was that the distribution of dimension $2n$ can admit only a integrable manifold of dimension n. Also it is true that every distribution of dimension $2n$ has at least an integrable manifold of dimension n. So this is the worst you can do.*

Lemma 10 *If $(M^{2m+1}, \alpha = 0)$ is a contact structure. Then the maximum dimension of the integrable submanifold is at most m.*

Proof: The proof is very simple. Suppose we have an involutive subdistribution ξ of $\alpha = 0$, then $d\alpha|_\xi$ is 0. This implies the dimension of distribution is at most n since $dimW + dimW^{d\alpha} = 2m$ and $W \subset W^{d\alpha}$ so $2dimW < 2m$. ■

Lemma 11 *If (M^{2m+1}, ξ) is a rank $2m$ distribution, then there always exists a integrable submanifold of dimension m.*

Proof: We are only looking locally so we consider a section of the normal bundle of the distribution say η. Now we define a form on the distribution given by $[X, Y] = \alpha(X, Y)\eta$. Now clearly this is alternating. Now if this is degenerate let κ be the Kernel of dimension $2k$. Then consider ξ/κ, then α is symplectic on this. So now we know there is a symplectic basis $e_1, e_2, ..., e_{2m-2k}$ such that α is zero on $e_1, .., e_{m-k}$. This precisely means the lie bracket lies in the distribution spanned by these vectors. Now adding κ we get that there is a involutive sub distribution of rank $m + k$. The worst case happens when $k = 0$ in which case the sub distribution will have rank only m. ■

Definition 12 *By a symplectic manifold (M,ω) we mean a manifold M together with a closed $2-$form ω such that TM is a symmmplectic vecttor bundle on M with respect to ω.*

A natural question to ask is:

When does a manifold admit onr of these two structures?

2 What is known

2.1 Contact case

Remark 13 *For a manifold M^{2m+1} to admit a contact structure, M must admit a co dimension 1 distribution D such that D can be given the structure of symplectic vector bundle. Note that we are not asking the form which gives symplectic structure to be $d\alpha$!!!*

Remark 14 *The above obstruction can be calculated purely algebraic topologically. Suppose the tangent bundle is characterised by the map $f : M \to BO(2m+1)$ then if we can get a lift $\tilde{f}$ as below we are done.*

$$\begin{tikzcd} & BU(m) \times BO(1) \arrow[d, "i"] \\ M \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & BO(2m+1) \end{tikzcd}$$

So this is a purely algebraic topology problem.

Now the question is: Is this sufficient?
Open Manifold: Yes!! proof is by h-principle.
Closed Manifold: Yes!! proof by Burman, Eliashberg, Murphy. Much harder.

2.2 Symplectic case

Remark 15 *For a manifold M^{2m} to admit a symplectic structure it must admit a $2-$form ω such that $\omega^n \neq 0$. Note we are not asking for the form to be closed. Now to solve this suppose TM is characterised my the map $f \to BO(2n)$ then if we get a lift to $BU(n)$ we are done.*

$$\begin{tikzcd} & BU(m) \arrow[d, "i"] \\ M \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & BO(2m) \end{tikzcd}$$

Now again one can ask is this sufficient?
Open Manifold: Yes it is. In fact we can actually get a symplectic structure with an exact form.
Closed Manifold: Here the situation is a bit more complicated. Since the form ω^n represents a non zero cohomology class and therefore ω represents a non zero

cohomology class. So there is an extra condition that the second cohomology is non zero.
So now we add another additional condition in this case that there exists $a \in H^2(M, \mathbb{R})$ such that $a^k \neq 0$ for all $k = 1, ..., n$. Now one can ask if this extra condition is sufficient? But still the answer is No!!! For example C Taubes showed that $\mathbb{C}P^2 \# \mathbb{C}P^2 \# \mathbb{C}P^2$ is a counterexample. No examples are known beyond dimension 4. So a popular belief is that it is actually true beyond 4.

3 h-principle for symplectic manifolds

Theorem 16 *Let M be an open almost symplectic manifold with form ω' which is not necessarily closed. Let a be some cohomology class. Then there exists a symplectic form ν on M with $[\nu] = a$.*

Proof: For the proof we first need a small observation.
Observation: *If we construct a form $\omega = d\alpha$ such that $d\alpha$ is symplectic. Then the theorem follows since if we take a ν which is closed and represents a, there exists a $M > 0$ large such that $d(M\alpha) + \nu$ is symplectic.*

Now the idea is to construct a form $d\alpha$ which is a holonomic section to approximate ω'. Then $d\alpha$ will automatically be non degenerate and it is exact so it is clearly closed.

Proof of the Approximation
Step 1: We want to show that there exists a map

$$\mathcal{S} : J^1(T^*M) \to \Lambda^2 T^*M$$

with the following properties

1. $\mathcal{S} \circ D\alpha = d\alpha$. Here $D\alpha$ denotes the first jet of α.
2. If we are given a $2-$form ω' there exists a $\tilde{\omega}' : M \to J^1(T^*M)$ such that $\mathcal{S}(\tilde{w}') = \omega'$.

This we will not prove globally since the final result we need is only local in nature but nevertheless the global result is true. Recall that locally we denoted $J^1(\mathbb{R}^n, \mathbb{R}^{n^*}) = J^1(T^*\mathbb{R}^n)$.
We know that $J^1(\mathbb{R}^n, \mathbb{R}^{n^*}) = \mathbb{R}^n \times \mathbb{R}^{n^*} \times Hom(\mathbb{R}^n, \mathbb{R}^{n^*})$. Now $T \in Hom(\mathbb{R}^n, \mathbb{R}^{n^*})$ can be regarded as a bilinear function

$$\tilde{T} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

given by $\tilde{T}(v_1, v_2) = T(v_1)(v_2)$
Now we define

$$\mathcal{S} : \mathbb{R}^n \times \mathbb{R}^{n^*} \times Hom(\mathbb{R}^n, \mathbb{R}^{n^*}) \to \Lambda^2(\mathbb{R}^{n^*})$$

given by $(x, y^*, T(v_1, v_2)) \to (x, T(v_1, v_2) - T(v_2, v_1))$ Now it is clear that $\mathcal{S} \circ D = d$. Now notice that the inverse image of a point is $\mathbb{R}^{n^*} \times F$. Here F is just the space of symmetric bilinear forms and hence is contractible.

Claim: Suppose $D^k \subset \mathbb{R}^n$ is a closed disc and suppose we are given

1. A smooth $2-$tensor $\tilde{\omega}'$ defined on $O_p(\partial D^k) \subset \mathbb{R}^n$
2. A $2-$form ω' on $O_p(D^k)$ such that $S(\tilde{\omega}') - \omega'$.

Then there exists a $2-$tensor $\tilde{\omega}'$ on $O_p(D^k)$ such that $S(\tilde{\omega}') = \omega'$. This extension follows from the fact that the homotopy groups of fiber are trivial.

Now this gives us a lift of ω' that is $\tilde{\omega}'$. Now we use holonomic approximation to approximate this by a $D\alpha$. Now we get $\mathcal{S} \circ D\alpha = d\alpha$ is close to ω'. Hence we have got the required $\omega = d\alpha$.

■

Theorem 17 *Let (M, ξ) be an open almost contact manifold. Then there exists a contact form ω on M which is close to the distribution ξ.*

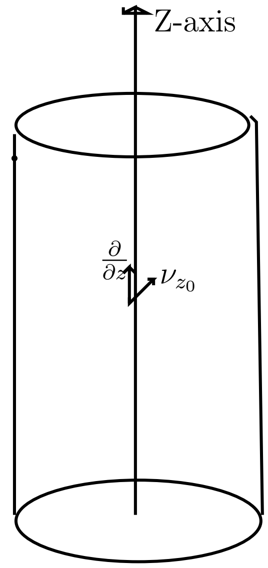
Proof: Now we have a form ω' corresponding to ξ and let α be a symplectic form on ξ. Now using $\mathcal{S}$ as in previous theorem we can define a map

$$J^1(T^*M) \to \Lambda^1(T^*M) \oplus \Lambda^2(T^*M)$$

where the second component is as before. and the first component is the value of the section. Now approximate (ω', α) by a holonomic section say $(\omega, d\omega)$. Now since it is close by it is non degenerate and therefore symplectic. ■

4 Expository Lecture on Contact Manifolds

Our basic question is: Given (M^3, ξ') where ξ' is any distribution we would like to homotope it to a contact distribution ξ.
But before we try and understand such general questions we will first try to understand the most basic contact structure. Take $I \times D^2 = (z, x, y)$



At each point (z_0, x, y) the distribution is spanned by $\frac{\partial}{\partial z}$ and ν_{z_0} which is a vector in the $x-y$ plane making an angle of z_0 with $x-$axis.
One can write down a formula for this $\frac{\partial}{\partial z}, \cos z_0 \frac{\partial}{\partial x} + \sin z_0 \frac{\partial}{\partial y}$ and the corresponding form is $\cos z_0 dy - \sin z_0 dx$

Theorem 18 *Locally every contact structure looks like this!!*

Proof: Let ξ_p be the distribution at p and let v_p be a vector field which is tangential to ξ_p and $v_p \neq 0$. This can be done since locally every distribution is locally a trivial distribution. Now rectify this vector field that is make it $\frac{\partial}{\partial z}$. Then in these new coordinates the form looks like $f(x,y,z)dx + g(x,y,z)dy$.
Now at the origin f or g is non zero. So we can assume without loss of generality that $f \neq 0$ in a small neighbourhood. Now we can write the form in a simpler form $dx + \frac{g}{f}dy$. Calling $H = \frac{g}{f}$ we have the form in the simple form $dx + Hdy$.
Until now we have not used the fact that the form is contact, so till here it is true for any distribution. Now the contact condition is $\alpha \wedge d\alpha \neq 0$.

$$d\alpha = \frac{\partial H}{\partial x} dx \wedge dy + \frac{\partial H}{\partial z} dz \wedge dy$$

So

$$\alpha \wedge d\alpha = \frac{\partial H}{\partial z} dx \wedge dz \wedge dy$$

So the condition is that $\frac{\partial H}{\partial z} \neq 0$. This geometrically means that the planes have to keep rotating. Now since $\frac{\partial H}{\partial z} \neq 0$ we have a change of coordinates

$$(x, y, z) \to (x, y, H(x, y, z))$$

Hence proved. ∎

Theorem 19 *Locally any contact structure is given by the* $Ker(\alpha)$ *where* $\alpha = dz - x_1 dy_1 - ... - x_n dy_n$

Proof: I know only a proof using Moser's trick as done by Yash have no clue how to set up the induction as Dishant suggested. ■

Now we come back to our previous example that is the contact structure on the cylinder. Notice that at $z = 0$ and $z = 1$ the distribution is the same. So it actually gives a contact structure on $S^1 \times D^2$. But $S^1 \times D^2 = S(T^*D^2)$.

We will interpret this differently. We are thinking of S^1 as set of rays passing through the origin in D^2. Now based on which point you are on the vertical S^1 pull back the corresponding ray on the D^2. The advantage of this interpretation is this if we can in some sense identify the S^1 in the tangent space of base with S^1 in the fiber we would have an invariant description which might carry over to a genus g surface. The way to do this put a Riemannian metric and take a $1-$form on S^1 and get the corresponding vector on base and now pull back this vector and consider the distribution spanned by the above vector and $\frac{\partial}{\partial t}$.

This is equivalent to this: At each point of $S(T^*D^2)$ just pull back the form corresponding to the form. This is the required contact form.

Theorem 20 *The above description gives a contact structure on* $ST^*\Sigma$ *where* Σ *is any genus* g *surface.*

We have produced a contact structure on $S(T^*S^2)$, which is same as $SO(3) = \mathbb{R}P^3$. But we have the covering map from $S^3 \to \mathbb{R}P^3$. Now pull back the contact form to get a contact structure on $\mathbb{R}P^3$.
Now every three manifold M is a branched cover of S^3 that is we have a covering projection

$$p : M - \cup \text{ links} \to S^3 - \cup \text{ closed curves}$$

Now the tubular neighbourhood of these links is a union of solid tori. Each solid torus has a contact structure on the boundary which we want to extend to the interior. Apparently there are some model structures which fit exactly. This is how contact structures were constructed in the $70'$s.

Q) Suppose a manifold M and a distribution D are given. Is it possible to find a contact structure on M which is homotopic to D as a plane field? What he did after this I did not understand so I will continue from where I understood that is the second approach to answer this question.

4.1 Approach 2

We know M^3/B^3 has a contact structure since it is open and h-principle holds. So now the natural question to ask is

Can any contact structure given on a neighbourhood of $\partial B^3 = S^2$ be extended?

So we need to understand the following questions

1. How many different contact structures are there in a neighbourhood of the boundary?

2. How many different contact structures are there on B^3?

We first need a definition

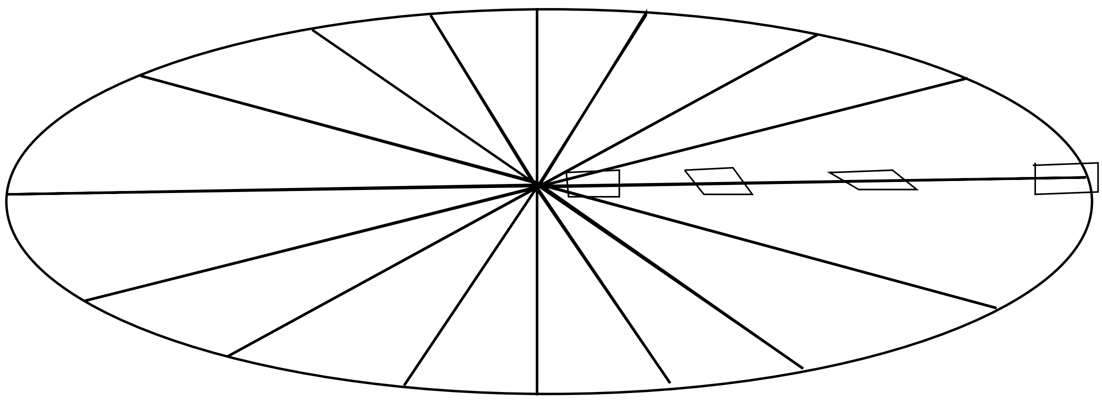
Definition 21 *Let* $\Sigma \subset M^{2n+1}$ *be a codimension* 1 *hypersurface then for every* $x \in \Sigma$ *we consider*

$$L_x = \{x \in T_x\Sigma \cap \xi : d\alpha(v, w) = 0 \text{ for all } w \in T_x\Sigma \cap \xi\}$$

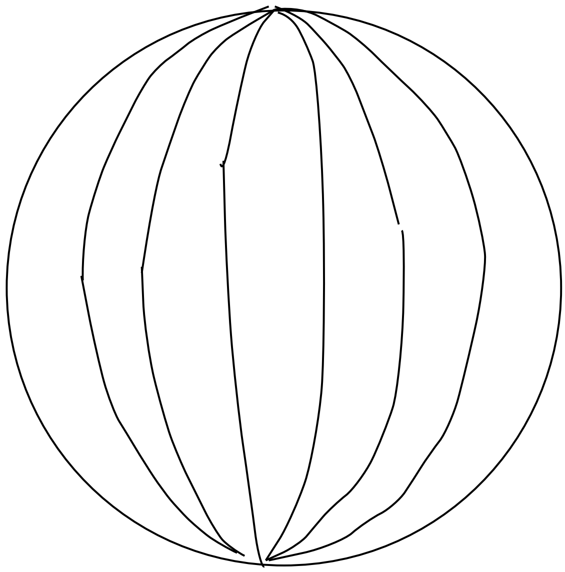
Now note that if we are unlucky then $d\alpha$ *might still be non degenerate and it is a disaster. But generically the intersection will be a* $2n-1$ *dimensional vector space which is odd and so* $d\alpha$ *has to have a kernel.*

Theorem 22 *The characteristic foliation determines the contact structure in a small neighbourhood.*

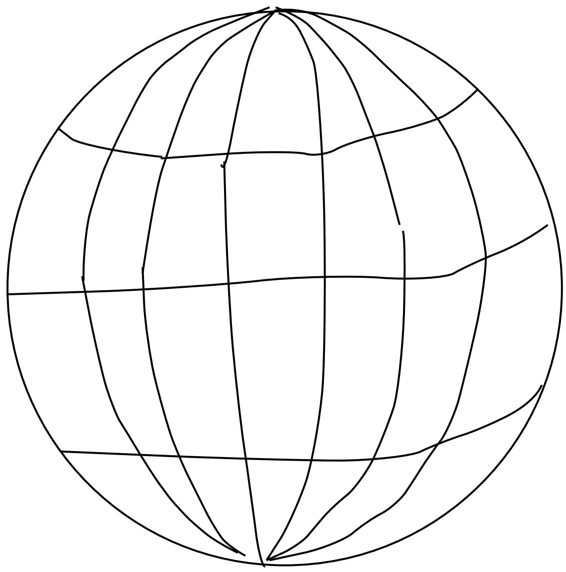
So the idea is to see what the characteristic foliation is and match it with the contact structure that produces that foliation. Now let me tell you something about the so called overtwisted disc.



So the contact structure is again $D^2 \times I$ but in the figure it is just depicted for the disc. The plane starts out horizontal at the center and then starts rotating as we move radially. These discs stacked up give the overtwisted disc. Now embed sphere in this in the usual manner, then it gives the charateristic foliation as shown below:



But this is not the only foliation possible by "jiggling" the sphere we can get more such foliations like shown below



The horizontal latitudes indicate the singularities because the contact plane is tangential to the sphere.

5 Yash's Lecture: Basics of Contact Geometry

First we will go through a few definitions quickly

Definition 23 *A contact structure ξ on a $2n+1$ manifold V is a codimension 1 hyperplane distribution at ξ is defined by $Ker(\alpha)$ for some $1-$form α satisfying $\alpha \wedge (d\alpha)^n \neq 0$.*

Remark 24 *Contact condition is equivalent to the condition $d\alpha|_\xi$ is non degenerate.*

Remark 25 *Contact condition only depends on $Ker(\alpha)$ that is if g is a non vanishing function then α is contact if and only if $g\alpha$ is contact.*

Remark 26 *Here the definition is actually for co orientable contact structure.*

Examples

1. $\mathbb{R}^{2n+1}$ with the form $\alpha = dz - \sum x_i dy_i$.
2. Contactization of Cotangent bundle: We know that there exists a canonical form λ_{can} a $1-$form on T^*M such that $d\lambda_{can}$ is the usual symplectic structure on T^*M. Then $(T^*M \times \mathbb{R}, dz - \lambda_{can})$ is a contact manifold.
3. (M, ω) is exact symplectic that is there exists a $1-$form λ such that $\omega = d\lambda$, then $(M \times \mathbb{R}, dz - \lambda)$ is contact.
4. Contact structure on co dimension 1 hyperplanes: We can identify this space with ST^*M. In this case the restriction of canonical $1-$form defines a contact structure on ST^*M.

Remark 27 *Check this a little problem is there!!!! Contact structures keep rotating along the legendrian: Let (V, ξ) be a contact manifold of dimension 3. We are choosing three here because in this case the contact condition is geometrically easy to understand. Let $p \in V$ and $\tilde{X}$ is a legendrian through p. Legendrian means a curve which is tangential to the distribution. It is clear that there are a lot of them locally by just integrating vector fields. We can choose a coordinate neighbourhood with coordinates (x, y, z) such that $\frac{\partial}{\partial z} = \tilde{X}'$. So the form looks like $dx + \tan\theta(x, y, z)dy$ where $\theta(x, y, z)$ is the angle made with X-axis. ow contact condition just means $\frac{\partial \theta}{\partial z} \neq 0$. This is precisely what we set out to prove.*

Remark 28 *Here rotating means with respect to nearby legendrians. That is if you go along a legendrian you might think it is not rotating but then you will see that just nearby you will see rotation on a close by legendrian. See the book suggested by Yash for the really nice pictures.*

5.1 Contact Stability

This section tells you that there are no local invariants. There are two levels at which one can make a statement.
At manifold level: Any two contact structures are equivalent.
At contact level: Locally any two contact structures can be homotoped to each other through contact structures.

Definition 29 *(V_1, ξ_1) and (V_2, ξ_2) are contactomorphic if there exists a diffeomorphism $\phi : V_1 \to V_2$ such that $\phi^*\xi_2 = \xi_1$*

Remark 30 *We do not require that the contact form pulls back to the contact form. If $\xi_1 = Ker(\alpha_1)$ and $\xi_2 = Ker(\alpha_2)$ then $\phi^*(\alpha_2) = f\alpha_1$ where $f \neq 0$.*

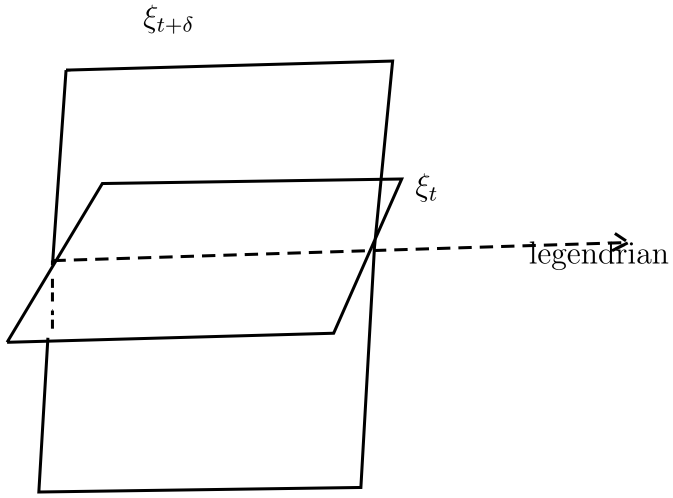
Definition 31 *We consider two contact structures ξ_0 and ξ_1 on V to be equivalent if there exists $\phi : V \to V$ a diffeomorphism such that $\phi^*\xi_1 = \xi_0$.*

Definition 32 *We say two contact structures ξ_0 and ξ_1 on V are isotopic if there exists an isotopy ϕ_t of V such that $\phi_1^*\xi_1 = \xi_0$.*

Remark 33 *If ξ_0 and ξ_1 are isotopic if there exists a family of contact structures $\{\xi_t\}$ on V joining ξ_0 and ξ_1 given by $\xi_t = \phi_t^*\xi_1$. Conversely we can ask: Given a family of contact structures $\{\xi_t\}$ does there exist a isotopy ϕ_t such that $\phi_t^*\xi_t = \xi_0$.*

5.2 Geometric Idea behind Moser's Trick

Let $\{\xi_t\}$ be a family of contact structures on V. Then try to construct "infinitesimal" vector fields which pull back $\xi_{t+\delta}$ to ξ_t. We will get a time dependent vector field X_t this way. The direction of this vector field can be found geometrically as follows:



Now along this legendrian the second vector field will rotate with respect to the first one and then we just have to make sure that the magnitude of vector field is correct.

5.3 Moser's Trick: The formal calculation

Let ϕ_t be an isotropy such that $\phi_t^*\xi_t = \xi_0$. Then this is equivalent to $\phi_t^*\alpha_t = f_t\alpha_0$ for some family of non vanishing functions $\{f_t\}$.

$$\frac{\partial}{\partial t}\phi_t^*\alpha_t = \frac{\partial}{\partial t}(f_t\alpha_0)$$

The right hand side is just $\frac{df_t}{dt}\alpha_0$. Now let us try to calculate the left hand side.

Theorem 34 $\frac{d}{dt}(\phi_t^*\alpha_t) = \phi_t^*(\frac{d}{dt}(\alpha_t) + L_{X_t}\alpha_t)$.

Proof: The proof is also taken from Anna Cannas Da Silva. Suppose $f(x,y)$ is a function of two variable then

$$\frac{d}{dt}f(t,t) = \frac{d}{dx}f(x,t)|_{x=t} + \frac{d}{dy}f(t,y)|_{y=t}$$

Now applying the above formula in our case we get

$$\frac{d}{dt}(\phi_t^*\alpha_t) = \frac{d}{dx}(\phi_x^*\alpha_t)|_{x=t} + \frac{d}{dy}(\phi_t^*\alpha_y)|_{y=t}$$

$$\frac{d}{dt}(\phi_t^*\alpha_t) = \phi_t^*(\frac{d}{dt}(\alpha_t) + L_{X_t}\alpha_t)$$

■

Now we get

$$\frac{df_t}{dt}\alpha_0 = \phi_t^*(\frac{d}{dt}(\alpha_t) + L_{X_t}\alpha_t)$$

$$\phi_{-t}^*(\frac{df_t}{dt}\alpha_t) = \frac{d\alpha_t}{dt} + L_{X_t}\alpha_t$$

Now we get

$$\frac{d\alpha_t}{dt} + L_{X_t}\alpha_t|_{\xi_t} = 0$$

Now we will use Cartan's magic formula

$$(\frac{d\alpha_t}{dt} + i_{X_t}d\alpha_t + di_{X_t}\alpha_t)|_{\xi_t} = 0$$

Now we choose $X_t \in \xi_t$ then $\alpha_t(X_t) \cong 0$. Then we get

$$\frac{d\alpha_t}{dt} + i_{X_t}d\alpha_t|_{\xi_t} = 0$$

This equation completely determines X_t since $d\alpha_t$ is non degenerate. Now integrate this and get ϕ_t. Now the ϕ_t satisfies

$$\frac{d}{dt}\phi_t^*\alpha_t = h_t\phi_t^*\alpha_t$$

for some family of functions h_t.

Theorem 35 ***Moser Trick*** *Let α_t be a family of contact forms and ξ_t the contact structures defined by them. Let X_t be a time dependent vector field characterised by*

1. $\alpha_t(X_t) = 0$
2. $i_{X_t}(d\alpha_t|_{\xi_t}) = -\frac{d\alpha_t}{dt}|_{\xi_t}$

Then $L_{X_t}\alpha_t + \frac{d\alpha_t}{dt} = h_t\alpha_t$.

Observation

1. If the flow $\{\phi_t\}$ exists for X_t for time large enough then

$$\frac{d\phi_t^*\alpha_t}{dt} = \gamma_t\alpha_0$$

   for some family of functions γ_t.

2. If for some $p \in V$, $\xi_t(p) \equiv \xi_0(p)$ then $X_t \equiv 0$.

Consequences

1. **Gray Stability**: ξ_t be a family of contact structures in a closed manifold V. Then there exists an isotopy ϕ_t of V such that $\phi_t^*\xi_t = \xi_0$.

2. **Relative Gray Stability**: Let ξ_t be a family of contact structure on closed manifolds such that $\xi_t = \xi_0$ on compact set A, then there exists a isotopy ϕ_t of V fixing A such that $\phi_t^*\xi_t = \xi_0$.

3. **Stability on Compact sets**: ξ_t be a family of contact structures on V such that $\xi_t \equiv \xi_0$ on A such that $A \subset V$ compact. Then there exists an isotopy $\phi_t : O_pA \to V$ such that $\phi_t^*\xi_t = \xi_0$.

Theorem 36 *Any contact structure is locally equivalent to standard contact structure on* $\mathbb{R}^{2n+1}$.

Proof: It is enough to show that any contact structure ξ_1 is a neighbourhood $O_p 0 \subset \mathbb{R}^{2n+1}$ is isomorphic to the standard contact structure on ξ_0. So we just want to construct a family of contact planes joining them
Technical point: WLOG we can assume $\alpha_1(0) = \alpha_0(0)$ and $d\alpha_0|_{\xi_0} = d\alpha_1|_{\xi_1}$. This implies $\alpha_t(0) = \alpha_0$ and $d\alpha_t|_{\xi_0}(0) = d\alpha_0|_{\xi_0}(0)$. α_t is contact at 0 so it is contact in the neighbourhood.
Technical Point: $\alpha_1(0) = \alpha_0(0)$ this can be done by just pushing α_1 with a linear isomorphism. Our problem reduces following linear algebra situation. V, f can be non zero element of V^* we have non degenerate bilinear form ω_1 and ω_2 on $W = Kerf$. To find an isomorphism of V say T such that $T^*f = f$ so $TKerf \simeq Kerf$ and $T^*\omega_2 = \omega_1$. ∎

5.4 Contact vector fields and Reeb vector fields

Definition 37 *We say X is contact if it's flow preserves the contact structure. If $\xi = Ker\alpha$ for a contact form α, this is equivalent to $L_X\alpha = h\alpha$*

Theorem 38 *On a contact manifold (V, ξ) given any vector field X, there exists a vector field X_ξ such that*

1. *$X_\xi \in \xi$*
2. *$X + X_\xi$ is contact.*

Proof: X is given. Let us fix contact form α such that $Ker\alpha = \xi$ for X_ξ such that $X_\xi \in \xi$. $X + X_\xi$ is contact if and only if $L_{X+X_\xi}\alpha = h\alpha$ if and only if $L_{X+X_\xi}\alpha|_\xi = 0$ if and only if $i_{X+X_\xi}d\alpha + di_{X+X_\xi}\alpha|_\xi = 0$ if and only if $i_{X_\xi}d\alpha|_\xi = -(i_X d\alpha di_X\alpha)|_\xi$. This shows X_ξ is uniquely determined.

∎

Remark 39

1. *Y is contact and legendrian implies $Y = 0$. Formal proof Y_ξ is unique so we can choose $Y_\xi = -Y$ or 0. This implies $Y = 0$. Y always rotate. So it better be zero.*

2. *We have a natural map contact vector fields $\to$ sections of TV/ξ. Observe that TV/ξ is trivial bundle. The trivialisation is given by*

$$TV/\xi \to V \times \mathbb{R}$$

$$(v, [x]) \to (v, \alpha(X))$$

Using this identify sections of TV/ξ with functions on V a section $[X] \to \alpha(X)$.

3. *Using 1 and 2 we have*

$$\textit{space of constant vector fields } \to C^\infty \textit{ functions on } V$$

$$X \to \alpha(X)$$

This map is clearly injective since $\alpha(X) = 0$ implies $X = 0$. For surjective choose $[X] \in \Gamma(TV/\xi)$ such that $\alpha(X) = \xi$. Now theorem implies there exists X_ϵ such that $X + X_\epsilon$ is contact.

Corollary 40 *Let X be a contact vector field on an open set $U \subset V$. Further suppose $K \subset U$ compact. Then there exists a contact vector field $\tilde{X}$ such that*

1. *$\tilde{X} \equiv X$ on a neighbourhood of K*

2. *$\tilde{X} \equiv 0$ outside.*

Proof: Multiply X with a bump function get X' which is 0 outside U and consider with X on a neighbourhood. Now apply the theorem such that X_ϵ such that $X' + X_\epsilon$ is contact. ■

Definition 41 *A vector field X is called Reeb if X is contact and X is transversal to ξ.*

Theorem 42 *Let (V, ξ) be a contact manifold. α a form defined by ξ. Let R_α be a vector field such that*

1. *$\alpha(R_\alpha) = 1$*

2. *$d\alpha(R_\alpha, -) = 0$*

Note that this uniquely determine R_α then R_α is a reeb vector field.

Proof: Since R_α is Reeb it preserves the contact structure. $L_{R_\alpha}\alpha = h\alpha$ if and only if $L_{R_\alpha}\alpha|_\xi = 0$ if and only if $i_{R_\alpha}d\alpha + di_{R_\alpha}\alpha|_\xi = 0$. Since $\alpha(R_\alpha) = 1$. The computation shows that $L_{R_\alpha}\alpha = 0$. ■

Remark 43 *Any Reeb vector field arises this way, that is if X is a reeb vector field then $X = R_\beta$ for some β such that $Ker(\beta) = \xi$.*

Now we make the correspondence between Reeb vector fields and functions stronger.

Theorem 44 *Let (V, ξ) be contact. Fix a contact form α such that $Ker(\alpha) = \xi$. Then there is a $1 - 1$ correspondence between contact vector fields and smooth functions on M given by*

$$X \to \alpha(X)$$

$$H \to X_H$$

given by $X_H = HR_\alpha + Y$ with $Y \in \xi$ satisfying $i_Y d\alpha|_\xi = -dH|_\xi$(or equivalently)