# GROUPS OF HOMOTOPY SPHERES

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Abstract. The aim of this article is to prove that exotic spheres in dimension $4k$ for $k > 1$ are finite. Historically exotic spheres appeared before the h-cobordism theorem so Milnor proves theorems without using this. But I have made use of h-cobordism theorem wherever it applies to shorten arguments.

## 1. Existence of Exotic Spheres

$D^4$ bundles[1] on $S^4$ are classified by $[S^3, SO(4)] \simeq \pi_3(SO(4)) \simeq \mathbb{Z} \oplus \mathbb{Z}$. Identifying $S^3$ with unit quaternions, a homotopy class corresponding to $(i, j)$, can be represented by a map $f \in [S^3, SO(4)]$ given by $f(x)y = x^i y x^j$. Denote the corresponding disc bundle by $\nu_{ij}$ and the sphere bundle by $\xi_{ij} := \partial\nu_{ij}$. If $E(\xi_{ij})$ is diffeomorphic to $S^7$ then we can attach $e^8$ to $\nu_{ij}$ and make it an $8-$manifold. Using the Gysin sequence we get

$$0 \to H^3(E(\xi_{ij})) \to \mathbb{Z} \xrightarrow{e\wedge} \mathbb{Z} \to H^4(E(\xi_{ij})) \to 0$$

and all other cohomology groups except $H^0$ and $H^7$ are zero. So, if the Euler class is $\pm 1$ then $E(\xi_{ij})$ has the homotopy type of a sphere. Since the Poincare conjecture[2] is true we know that it is homeomorphic to the sphere.

**Lemma 1.** $e(\nu_{ij}) = i + j$

*Proof.* The proof will be given later. □

Using lemma 1, we know that $E(\partial\nu_{ij})$ is homeomorphic to $S^7$ whenever $i + j = 1$. Now suppose that we get the standard sphere whenever $i + j = 1$. Then we can attach a $e^8$ to $E(\nu_{i,1-i})$ along the boundary and get an $8-$manifold $M_{i,1-i}$.

**Lemma 2.** $p_1(M_{i,1-i}) = 2(2i - 1)$

*Proof.* The proof will be given later. □

We can choose an orientation on $M_{i,1-i}$ such that the sign of the intersection form is $+1$. Now we can calculate $p_2(M_{i,1-i})$ using the Hirzebruch Signature formula

$$p_2(M_{i,1-i}) = \frac{p_1^2(M_{i,1-i}) + 45}{7}$$

When $i = 2$ we get $p_2(M_{i,1-i}) = \frac{81}{7}$, which is impossible! Since it has to be an integer. This means the manifold $M_{i,1-i}$ does not exist. So the boundary is not the standard sphere and is therefore exotic!

*Proof of Lemma 1.* We know that the Euler class is the obstruction cocycle to finding a non zero section. To find a section we just need to find sections

$$s_1 : D^4_+ \to D^4_+ \times S^3$$
$$s_2 : D^4_- \to D^4_- \times S^3$$

such that on the identification they are the same. Since both the bundles are trivial we can construct a section on the three skeleton $S^3 = D^4_+ \cap D^4_-$ by taking $s_1(x) = (x, 1)$ and $s_2(x) = (x, x^{i+j})$. Therefore the obstruction cocycle to extending it to $D^4_+ \cup D^4_-$ is exactly $i + j$. ■

[1]Milnor was led to consider these while he was trying to classify $n - 1$ connected $2n$ manifolds

[2]Milnor was not aware of this, so he initially believed that he disproved the Poincare conjecture

*Proof of Lemma 2.* Before we calculate the Pontryagin class of $M$ we will first calculate the Pontryagin class of $\nu_{i,j}$. First notice that Pontryagin class is linear in $i$ and $j$ since it is just the pull back of a class in $BSO(4)$ and a class is just represented by a map $\phi : BSO(4) \to K(Z,4)$. So the map corresponding to $f+g$ is

$$S^4 \to S^4 \vee S^4 \xrightarrow{f \vee g} BSO(4) \xrightarrow{\phi} K(Z,4)$$

Clearly this is the same as

$$S^4 \to S^4 \vee S^4 \xrightarrow{f \circ \phi \vee g \circ \phi} K(Z,4)$$

This shows that it is linear in $i$ and $j$. Furthermore we know that it is independent of the orientation of the fibre. But if the orientation is reversed then $\nu_{i,j}$ gets replaced by $\nu_{-j,-i}$. So the Pontryagin class is given by an expression $c(i-j)$. Now by actually calculating the Pontryagin class in a special case we get $c=2$. So we get that the Pontryagin class of $v_{i,j} = 2(i-j)$.
Now we calculate the Pontryagin class of $\tau(v_{i,j})$. We know that $\tau(v_{i,j}) = \tau(S^4) \oplus v_{ij}$. By the Whitney sum formula we get that the Pontraygin class is $2(i-j)$. Now $H^4(M_{i,1-i}) \to H^4(\nu_{i,1-i})$ is surjective and we know that pullback of the Pontryagin class of $\tau(M_{i,1-i})$ is $2(2i-1)$. So the Pontryagin class of $M_{i,1-i}$ is precisely $2(2i-1)$. ■

## 2. Exotic spheres as h-cobordism classes

**Definition 3.** *A manifold $M$ is said to be a homotopy $n$-sphere if $M$ is closed and has the homotopy type of the sphere $S^n$.*

**Definition 4.** *Two closed manifolds $M_1$ and $M_2$ are h-cobordant if the disjoint sum $M_1 + (-M_2)$ is the boundary of some manifold $W$, where both $M_1$ and $-M_2$ are deformation retracts of $W$.*

**Remark 5.** *h-cobordism is an equivalence relation.*

**Theorem 6.** *The h-cobordism classes of homotopy $n$-spheres form an abelian group under the connected sum operation. We denote this group as $\Theta(n)$*

Now we will relate this to the exotic spheres using the h-cobordism theorem

**Theorem 7.** *(Smale) If $M_1, M_2$ and $W$ are simply connected and the dimension of $M_1$ and $M_2$ is at least 5. Then $W$ is diffeomorphic to $M_1 \times I$. In particular $M_1$ and $M_2$ are diffeomorphic.*

**Remark 8.** *Theorem 7 implies that diffeomorphism classes of spheres are in $1-1$ correspondence with h-cobordism classes of homotopy spheres.*

We will now identify the h-cobordism class of the standard $n$-sphere.

**Lemma 9.** *A simply connected manifold $M$ is h-cobordant to the sphere $S^n$ if and only if $M$ bounds a contractible manifold.*

*Proof.* If $M$ is h-cobordant to $S^n$, then by h-cobordism theorem we know that $M$ is diffeomorphic to $S^n$ and therefore clearly bounds a contractible manifold.
Conversely if $M = \partial W$ with $W$ contractible, then removing the interior of an imbedded disk we obtain a simply connected manifold $W'$ with $\partial W' = M + (-S^n)$. By excision we know that the map $(D^{n+1}, S^n) \hookrightarrow (W, W')$ is an isomorphism. Now using the long exact sequence in homology we get that $S^n \hookrightarrow W'$ is an isomorphism on homology; hence $S^n$ is a deformation retract. Therefore $M$ is h-cobordant to $S^n$.

□

## 3. Outline of the proof

**Lemma 10.** *A homotopy n-sphere is S-parallelisable.*

*Proof.* The proof uses Adam's theorem on J-homomorphism and is given in the later section. □

**Lemma 11.** *Let $M$ be an n-dimensional submanifold of $\mathbb{R}^{n+k}$, $n < k$. Then $M$ is s-parallelisable if and only if it's normal bundle is trivial.*

*Proof.* The proof is based on the following lemma □

**Lemma 12.** *If $\xi$ is a rank $k$ vector bundle over an $n$ dimensional CW complex $X$ such that $k > n$. Then $\xi$ is trivial iff $\xi \oplus \varepsilon^1$ is trivial*

*Proof.* Using the long exact sequence associated to the fiber sequence

$$S^k = O(k+1)/O(k) \xrightarrow{j} BO(k) \xrightarrow{i} BO(k+1)$$

We have an exact sequence of sets

$$[X, S^k]_* \xrightarrow{j_*} [X, BO(k)]_* \xrightarrow{i_*} [X, BO(k+1)]_*$$

If $f$ is the classifying map for $\xi$ we have that $i \circ f$ is nullhomotopic.

This shows that $i_*[f] = [0]$, so there exists $g\colon X \to S^k$ pointed such that $[f] = j_*[g] = [jg]$. Since the dimension of $X$ is smaller than $k$, the map $g$ is nullhomotopic by cellular approximation. Hence $f$ is nullhomotopic and so $\xi$ is trivial. □

*Proof of Lemma 11.* Let $\tau$ and $\nu$ be the tangent and normal bundles. Then $\tau \oplus \nu$ is trivial. Hence $(\tau \oplus \varepsilon^1) \oplus \nu$ is trivial. Therefore by Lemma 12, $\nu$ is trivial. ■

Combining Lemma 10 and Lemma 11, we have that the normal bundle of a homotopy n-sphere is trivial. Hence the normal bundle can be framed say $\phi$. Allowing the normal frame field $\phi$ to vary we get

$$p(M) = \{p(M, \phi)\} \subset \pi_n^s$$

**Lemma 13.** *The set $p(S^n) \subset \pi_n$ is a subgroup and for any homotopy $n-$sphere $\Sigma$, $p(\Sigma)$ is a coset. Thus the map $\Sigma \to p(\Sigma)$ defines a homomorphism from $p' : \Theta(n) \to \pi_n/p(S^n)$.*

*Proof.* We will use the fact that $p(M) + p(M') \subset p(M\#M')$. The proof of this is rather technical. Combining this with the identities

(1) $S^n \# S^n = S^n$
(2) $S^n \# \Sigma = \Sigma$
(3) $\Sigma \# -\Sigma = S^n$

we get

$$p(S^n) + p(S^n) \subset p(S^n)$$

this implies $p(S^n)$ is indeed a subgroup

$$p(S^n) + p(\Sigma) \subset p(\Sigma)$$

this implies that $p(\Sigma)$ is a union of cosets

$$p(\Sigma) + p(-\Sigma) \subset p(S^n)$$

this implies that $p(\Sigma)$ is a single coset. □

To prove that $\Theta(n)$ is finite we now just need to show that kernel of $p'$ is finite. The following lemma gives a characterisation of the kernel which we will use to prove the theorem.

**Lemma 14.** *$p'(\Sigma) = 0$ iff $\Sigma$ bounds a parallelisable manifold.*

*Proof.* If $\Sigma = \partial W$ then using Whitneys's theorem on embeddings, we can extend the embedding of $\Sigma \subset \mathbb{R}^{n+k}$ to an embedding of $W \subset D^{n+k+1}$. Let $\psi$ be a framing of $W$, then $\psi|_M$ is a framing of $\Sigma$. This framing clearly gives a map $S^{n+k} \to S^n$ which is null homotopic.
Conversely if $p(\Sigma, \phi) = 0$, then by the Pontryagin Thom theorem $\Sigma$ bounds a manifold $W \subset D^{n+k+1}$, where $\phi$ extends to a framing $\psi$ of $W$. Lemma 11 implies that $W$ is s-parallelisable. It is a manifold with boundary so it is actually parallelisable. □

**Theorem 15.** *If a homotopy sphere of dimension $4k$ bounds a parallelisable manifold then it bounds a contractible manifold.*

*Proof.* This is the main theorem and the last section is just dedicated to proving this statement. □

## 4. Homotopy spheres are S-parallelisable

For the proof we need Adam's theorem about the J-homomorphism.

**Theorem 16.**
1. *If $n \equiv 0 (mod\ 8)$, the image of $J$ is $\mathbb{Z}/2$.*
2. *If $n \equiv 1 (mod\ 8)$, the image of $J$ is $\mathbb{Z}/2$.*
3. *If $n \equiv 3 (mod\ 4)$, the image of $J$ is $\mathbb{Z}/m(2s)$. (Where $n = 4s - 1$ and the function $m$ is related to the denominators of Bernoulli numbers)*

Let $\Sigma$ be a homotopy sphere. We know that the obstruction to triviality of $\tau \oplus \varepsilon^1$ is a well defined cohomology class

$$\mathfrak{o}_n(\Sigma) \in H^n(\Sigma; \pi_{n-1} SO_{n+1}) = \pi_{n-1}(SO(n+1))$$

This coupled with lemma 12 gives us that this determines an element $\mathfrak{o}_n^s$ of $\pi_{n-1}(SO)$ and $\mathfrak{o}_n^s = 0$ iff $\mathfrak{o}_n = 0$.
1. If $n \equiv 3, 5, 6, 7 (\text{mod } 8)$ then $\pi_{n-1} SO = 0$.
2. If $n \equiv 0, 4 (\text{mod } 8)$ then Kervaire[1] proved that $\mathfrak{o}_n = c_1 p_k(\tau \oplus \varepsilon^1) = c_1 p_k(\tau)$. But by Hirzebruch signature formula we have $p_k = c_2 \sigma = 0$, where $\sigma$ is the signature of $\Sigma$.
3. If $n \equiv 1, 2 (\text{mod } 8)$, it follows from an argument of Rohlin that $J(\mathfrak{o}_n) = 0$. But Adam's theorem implies $J$ is injective so $\mathfrak{o}_n = 0$.

## 5. Proof of Theorem

Let $\phi : S^p \times D^{q+1} \to M$ be an embedding. Let $M'$ be obtained by doing surgery using $\phi$. The following lemma tells the effect of surgery on homotopy groups

**Lemma 17.** *The homotopy groups of $M'$ are given by*

$$\pi_i M' \simeq \pi_i(M) \qquad \text{for } i < min(p, q)$$

$$\pi_p(M') \simeq \pi_p(M)/\Lambda \qquad \text{if } p < q$$

*where $\Lambda$ denotes a subgroup generated by $\lambda = \phi(S^p \times 0)$*

**Lemma 18.** *If $M^n$ is parallelisable and if $p < \frac{n}{2}$, then any class $\lambda$ is represented by some embedding $\phi : S^p \times D^{n-p} \to M$.*

*Proof.* Whitney's theorem implies that $\lambda$ can be represented by an embedding.Now $M^n$ is parallelisable so the normal bundle is trivial. Hence there is an embedding $\phi : S^p \times D^{n-p} \to M$. □

We now specialise to the case of $n = 4k + 1$ since we want to show that $\Theta(4k)$ is finite. By doing surgery as above we can kill homotopy groups up to dimension $2k - 1$. In dimension $2k$ if we try to do surgery we have to replace $S^{2k} \times D^{2k+1}$ replaced by $D^{2k+1} \times S^{2k}$. So we might introduce a homotopy group in the same dimension. The rest of the proof is to understand the effect of surgery in middle dimension.

**Lemma 19.** *Let $M'$ be obtained from $M$ by surgering $\phi : S^{2k} \times D^{2k+1}$. Then*

$$\frac{H_k(M)}{\langle\lambda\rangle} \simeq \frac{H_k(M')}{\langle\lambda'\rangle}$$

*where $\lambda$ corresponds to $\phi(S^{2k} \times 0)$ and $\lambda'$ corresponds to $\phi(s_0 \times S^{2k})$.*

Suppose we choose an element $\lambda \in H_k(M)$ such that $\mu.\lambda = 1$ for some $\mu \in H_{k+1}(M)$ then using the long exact sequence in homology

$$H_{2k+1}M \xrightarrow{\cdot\lambda} H_{2k+1}(M, M - \text{Im } \phi) \to H_{2k}(M - \text{Im } \phi) \to H_{2k}(M) \to 0$$

Since the left most map is surjective we get that

$$H_{2k}(M - \text{Im } \phi) \simeq H_{2k}(M)$$

This implies that $\lambda' = 0$. Thus we have

$$\frac{H_k(M)}{\langle\lambda\rangle} \simeq H_k(M')$$

Therefore by Poincare duality we can kill all the non torsion part using surgery. Now we claim that if we do a surgery on $M$ whose homology has just torsion, then the betti number changes and the torsion subgroup becomes smaller. Then we can do surgery using the primitive class and get a manifold with strictly smaller $H_k$.

**Lemma 20.** *Let $M$ be a $4k+1$ manifold with non zero homology only in middle dimensions and such that $H_{2k}$ is only torsion. Then a surgery changes the $2k$ betti number.*

*Proof.* Let $W$ be a cobordism between $M$ and $M'$. Define the semi-characteristic class

$$e^*(\partial W) = \sum_{i=0}^{2k} H_i(\partial W)(\text{mod } 2)$$

Note that the summation is only on half the homology groups. So if $e^*$ changes then we know that the rank of $H_{2k}$ changed. Also we know that $e(W) = e(M) - 1 = -1$, since it is just $M \times I$ with a cell attached and $e(M) = 0$. It can be shown that the rank of intersection pairing

$$H_{2k+1}W \times H_{2k+1}W \to \mathbb{R}$$

is $e^*(\partial W) + e(W)(\text{mod } 2)$. But we know this pairing is skewsymmetric and therefore has even rank. So we have

$$e^*(M + M') + 1 \equiv 0(\text{ mod } 2)$$

This proves that the rank of $H_{2k}$ changes. □

From lemma 19 we know that

$$\frac{H_k(M)}{\langle\lambda\rangle} \simeq \frac{H_k(M')}{\langle\lambda'\rangle}$$

Since the rank changes and $\langle\lambda\rangle$ is finite it follows that $\langle\lambda'\rangle$ is non torsion. Hence $H_k(M')$ has strictly smaller torsion. Now killing the new primitive element we get a manifold with strictly smaller $H_{2k}$.

## References

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[5] John Milnor. *Lectures on the h-cobordism theorem.* Princeton University Pres, 2016.