Shape of space

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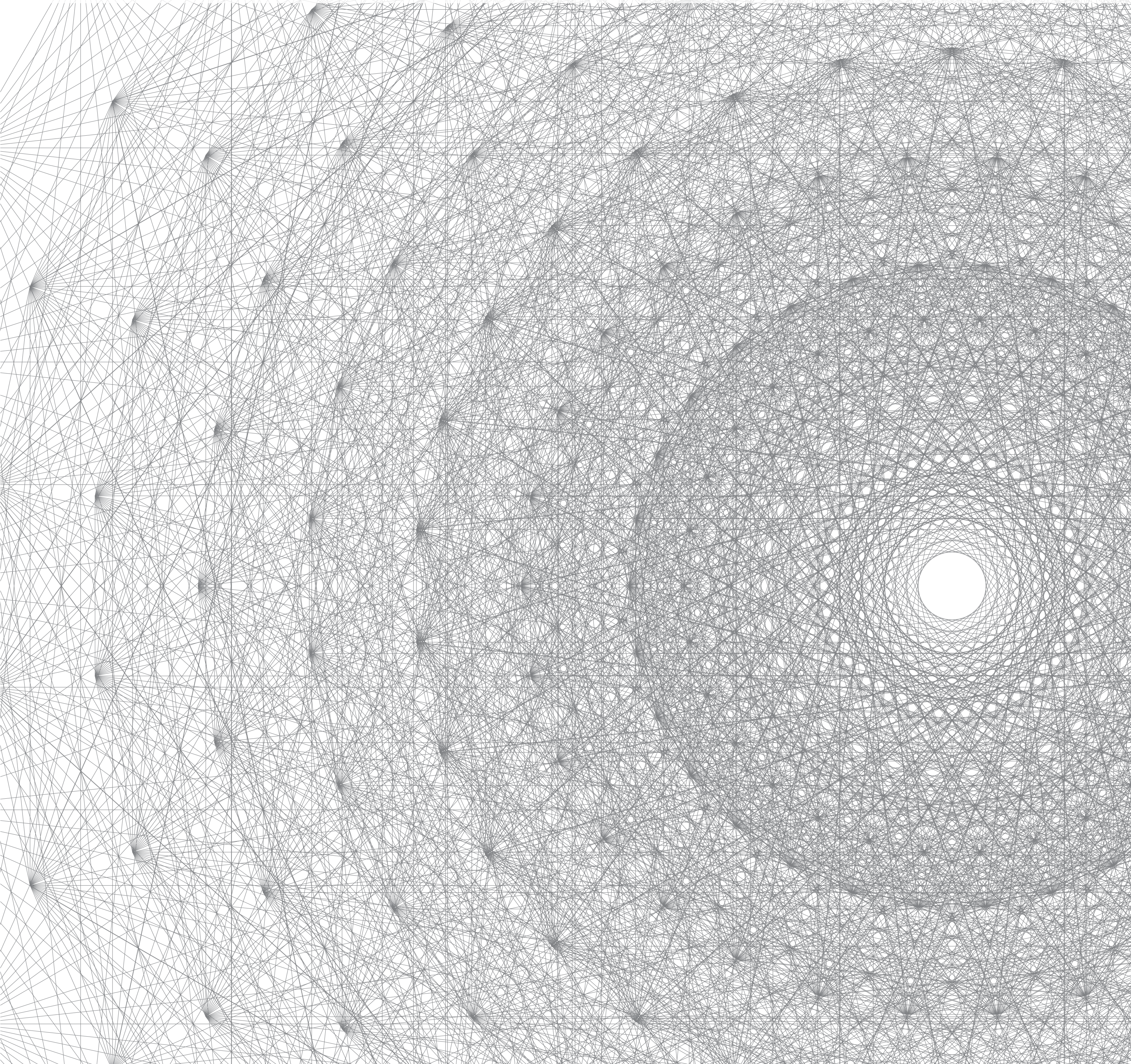
Sim Camp, Summer   2021



Acknowledgements

The main idea for these lecture series and notes is taken from the wonderful book "Shape of space" by Jeffrey Weeks which I strongly recommend. A lot of material in these notes is taken from there. As for the images I have borrowed almost all of them from either google or youtube. Lastly, these notes wouldn't have been possible without the funding and support I received from all my colleagues at SIM camp. They patiently went through the videos and notes I made and helped me improve the material. I would also like to thank my girlfriend Maria, for making me all the tasty food while I was typing these notes.

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CHAPTER 1

Introduction

sec:
intro

For thousands of years mankind has been trying to work out the shape and structure of the universe we live in. Is the earth flat or round? Do the planets orbit around the Earth or the Sun? What shape are their orbits? Where are the stars in relation to the planets? Mathematics has always been crucial for this endeavour. Much of the history of astronomy concerns the use of the most basic shapes - circles, spheres etc - and yet people have disagreed enormously because of our inability actually to see what shape things are. So how do we decide the answers to these questions when we cannot see it? Historically there were some very interesting ideas but they used a lot of complicated mathematics! We, armed with the power of insight can answer these questions much better.

1.1 Ancient flat earth theories

Many ancient cultures believed that the earth was flat. For example, Ancient babylonians(earlier than the 9th century BC) believed that world was a disc, surrounded by a ring of water called the "Bitter River".

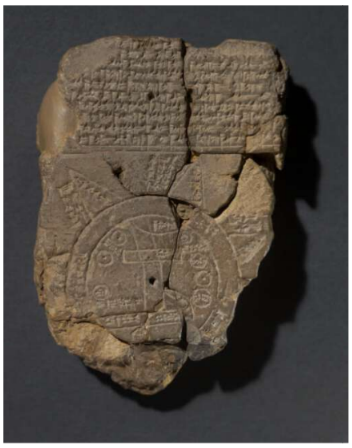

Figure 1.1: Babylonian map of the Earth

1.2 The spherical earth

Even though the ancient people believed that the earth was flat, by the time the Roman writer Pliny the Elder was writing the first part of his Natural History around AD 77, the fact that the Earth is a sphere was treated as common knowledge: 'We all agree on the earth's shape. For surely we always speak of the round ball of the Earth'.

Figure 1.2: Pliny: Natural History

1.3 How did they know?

So from history we know that as early as 77 AD people knew that the earth was a sphere, but 9 centuries before that they thought it was flat. So how did they figure it out?

Phoenicians

In The Histories, written 431–425 BC, Herodotus stated that Phoenician explorers during their circumnavigation of Africa, found that the Sun was not above them but to their right. If the Earth is flat, then the Sun should always be above you.

Aristotle

Aristotle (384–322 BC) believed that the Earth was a sphere based on his observations of ships with tall masts moving across the horizon. If the Earth were flat then the whole ship would get smaller as it moved further away. As it moved further, you would be able to see the whole of the ship getting smaller.

Figure 1.3: Ship at the horizon

Aristotle observed that when ships sailed over the horizon the bottom part of a ship, the hull, actually disappeared from view. If it moved further the less of the ship you could see – this could only happen if the Earth was spherical.

Magellan and circumnavigation of the earth

Magellan first set sail in September 1519 as part of an epic attempt to find a western route to the spice-rich East Indies in modern-day Indonesia.In September 1522, one of his ships arrived safely back in Spain having completed a successful circumnavigation of the globe. If the Earth were flat then they would have reached its edge eventually.

1.4 Intrinsic versus Extrinsic

So far we have seen three reasons given for figuring out the shape of the earth. But you can categorize these three reasons into types. One type is “Intrinsic” whereas the other ”extrinsic”.

What is extrinsic?

In this particular case, we were interested in the shape of earth. So ideally our reasoning should some how only think about the object earth and figure out it’s shape. But if you take the reasoning of the Phoenicians or Aristotle you will notice that they use external objects like the sun or the ship to figure out something about the earth. We call this sort of reasoning extrinsic reasoning. We will not be interested in this sort of reasoning because if we want to apply similar techniques to figuring out the shape of universe, we need to find something external to our universe which is not possible at present.

What is intrinsic?

In the case of the example of circumnavigation, we are reasoning by talking about a path on the earth which would not have existed if the earth were round. So we are just directly studying this object. So this is called intrinsic. We will be interested in these sorts of intrinsic reasonings!

CHAPTER 2

Flatland

sec: second

We will start by watching the video. Long long ago in a world very different from our own, there lived an ordinary individual called the square. Square as you might expect is actually just a square. He lived in a world called the Flatland. The flatland was a two dimensional universe and all the residents of the Flatland aka Flatlanders only moved in two directions and had no idea that there could be another dimension. They believed that the Flatland was a giant plane. To be precise, they assumed it was a plane because for as long as they could see it was just a flat plane. The inhabits of the flatland were lines, triangles, squares and more sided polygons. As the sides got larger they became closer and closer to a circle. Whenever these flatlanders meet, due to the two dimensional nature of their universe, they just see the one dimensional projection similar to how you just see the two dimensional projection of me on the screen.So all flatlanders look like a line to each other. So how do they distinguish one from another? They sing

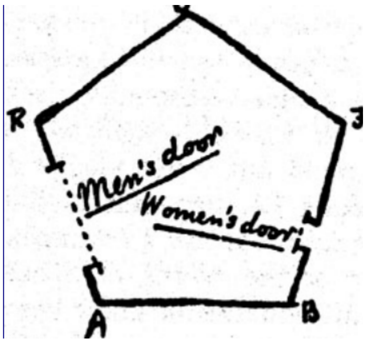

Figure 2.1: House in Flatland

Once a physicist Ms Stone, published that the Flatland could possibly be curved onto itself and compared it to a circle. Of course everyone dismissed her theories because they reasoned the Flatland was infinite and clearly flat so it was nothing like a circle. Needless to say no one followed this any further except for our ordinary protagonist Square who found this very interesting.

Figure 2.2: Shape of our universe?

2.1 Expeditions of a Square

First expedition

In order to prove that the flatland was curved, Square decided to travel east and by just continued to travel east wanted to end up at the same point. After travelling for three days square ended up at the same point. But of course

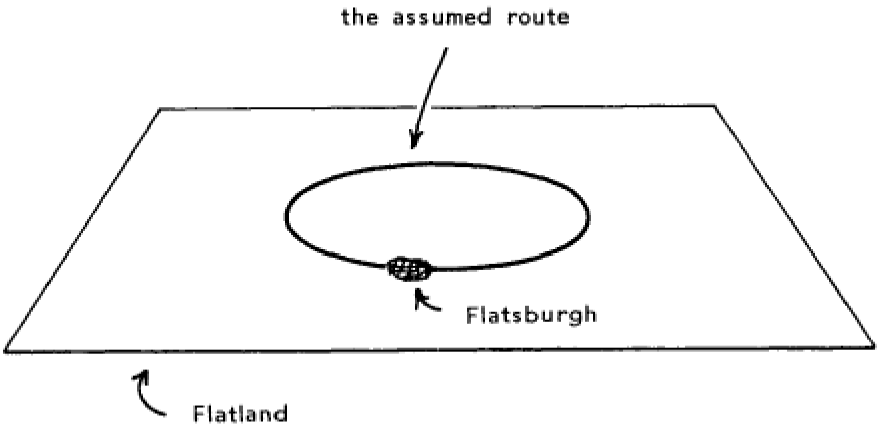

Figure 1.2 Even A Square's companions thought they had veered in a circle.

Square could have just traversed a big circle. So this time square set out with hexagon to make sure they go in a straight line. After travelling for three days they ended up exactly where they started.

Second expedition

To really prove that something weird was happening, they decided to go in perpendicular directions and they again ended up at the same point but square who travelled north took longer and never met the hexagon who travelled east. What are the possible shapes for the flatland universe?

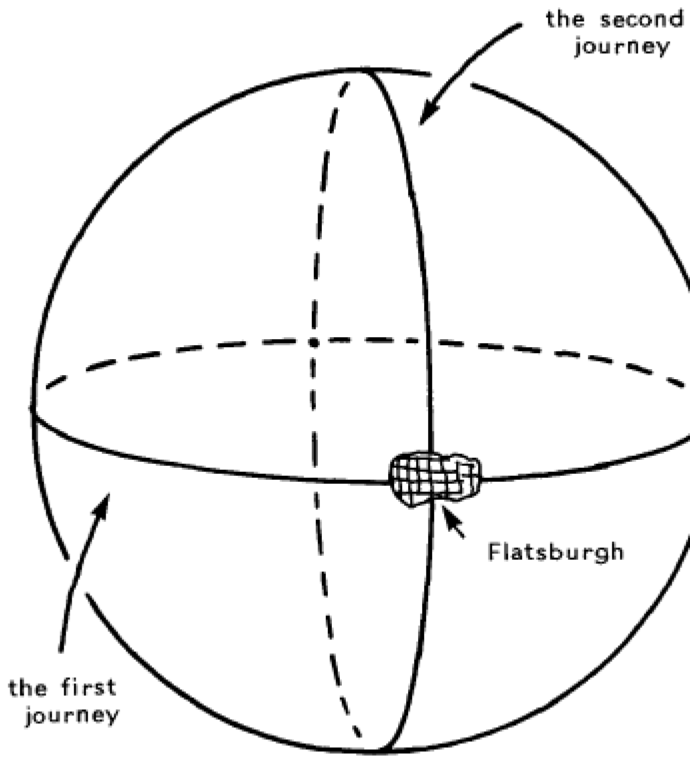

Figure 1.3 The two threads ought to cross, even if Flatland were a "hypercircle" (i.e., a sphere).

CHAPTER 3

Topology

sec: third

Topology is a branch mathematics, concerned with shapes of objects. Here we don't really care too much about the exact shape. For example when we say the Earth is sphere, we mean it is an object obtained from the sphere by some deformations.

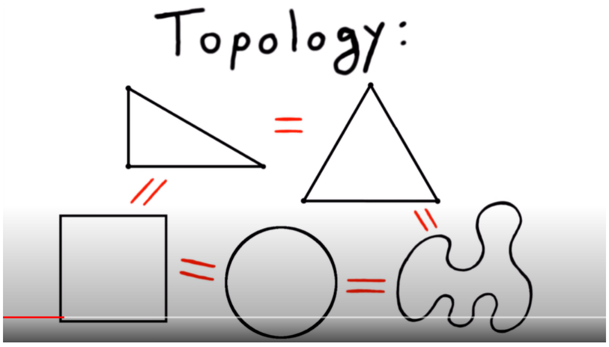

Figure 3.1: Allowed deformations

But we are not allowed to do any deformations. For example, you cannot cut it or glue together different parts. Similarly we don't allow gluing!

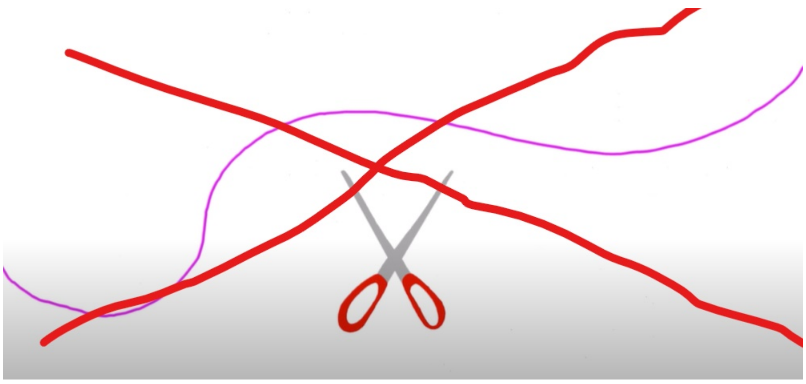

Figure 3.2: Cutting not allowed

So informally we can think of this as studying squishy shapes or studying shapes made with rubber.

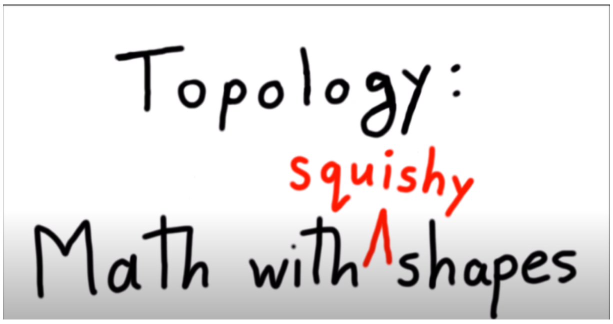

Figure 3.3: Topology in a nutshell

Due to the fact that shapes are allowed to be squished many shapes which look very different can be the same!
For example the two shapes below are the same! So how do we distingush???

Figure 3.4: Shape 1

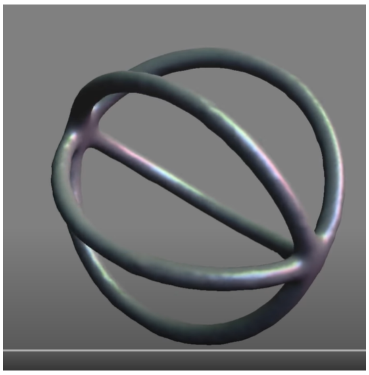

Figure 3.5: Shape 2

We need invariants

CHAPTER 4

First Invariant: Euler Characteristic

sec: fourth

4.1 What are invariants?

Before we define the Euler characteristic, we need to understand what an invariant is. Invariants are some "algebraic objects" which we associate to spaces. For example it could be a number like dimension. But if two spaces are the same, the invariant has to be the same. Using invariants we can distinguish spaces!

4.2 Euler Characteristic

To define the Euler characteristic we need to first break the object into triangles. After breaking it into many triangles, we will count the number of vertices, edges and faces. Let V be the number of vertices, E the number of edges and F the number of faces. We define the Euler characteristic to be the number $V - E + F$.

Theorem 4.2.1. *(Classification of two dimensional compact oriented surfaces) The only two dimensional compact oriented surfaces are given by sphere, torus and their connected sums.*

If the theorem above looks too scary, all that it is saying is that the only possibilities for surfaces are sphere, torus, double torus, triple torus and so on.

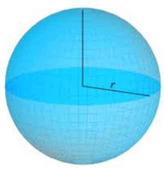

Figure 4.1: Sphere

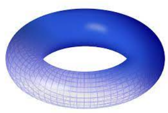

Figure 4.2: Torus

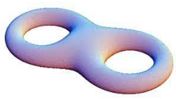

Figure 4.3: Double torus

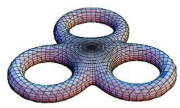

Figure 4.4: Triple torus

Euler characteristic distinguishes between all these surfaces!

4.3 Computing Euler characteristic for a sphere

A sphere has many models. Euler characteristic is an invariant so, it should give the same answer on all these models.

Tetrahedron

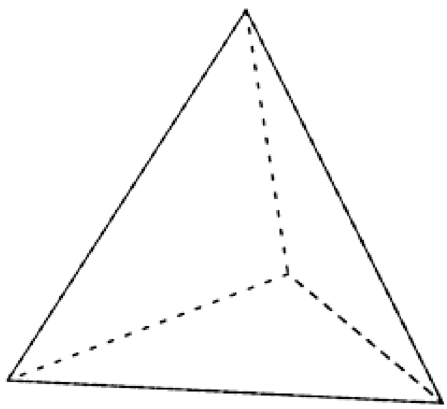

Figure 4.5: Tetrahedron

The number of faces F are 4, the number of edges are 6 and the number of vertices are 4. Therefore the Euler characteristic is $4 - 6 + 4 = 2$.

Octahedron

Ex:*Calculate the Euler characteristic.*

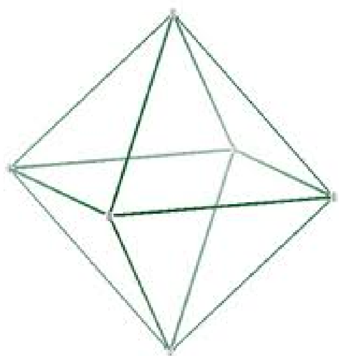

Figure 4.6: Octahedron

Icosahedron

Ex:*Calculate the Euler characteristic.*

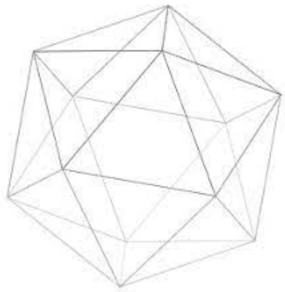

Figure 4.7: Icosahedron

CHAPTER 5

Gluing

Gluing is a very useful technique to construct difficult spaces from more familiar spaces. We will illustrate this with an example.

5.1 Cylinder

We will construct a cylinder from a square by gluing. This would be a nice way for a flatlander to visualize a cylinder. Flatlanders cannot see three dimensions, so all they can do is put together two dimensional things to try and understand the higher dimensions.

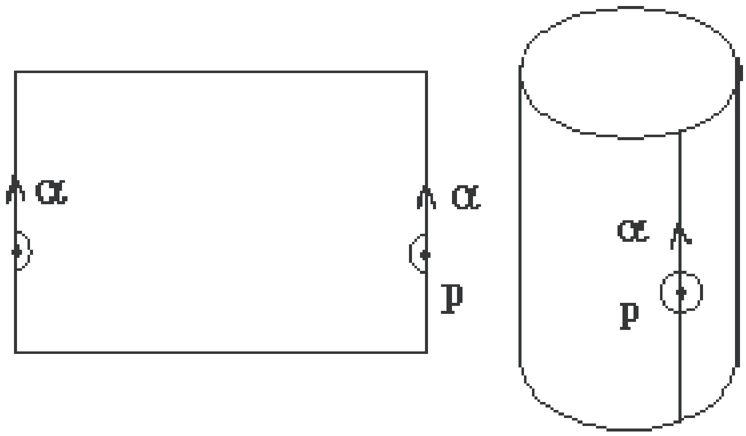

Figure 5.1: Gluing to make a cylinder

As in the figure we will be using arrows along the edges to represent sides we will glue together.So we can think of a cylinder as the space obtained by attaching the two sides with glue. Observe that the half circles around the point P glue together into a nice circle.

5.2 Torus

Now we will come up with a similar gluing model for a torus.

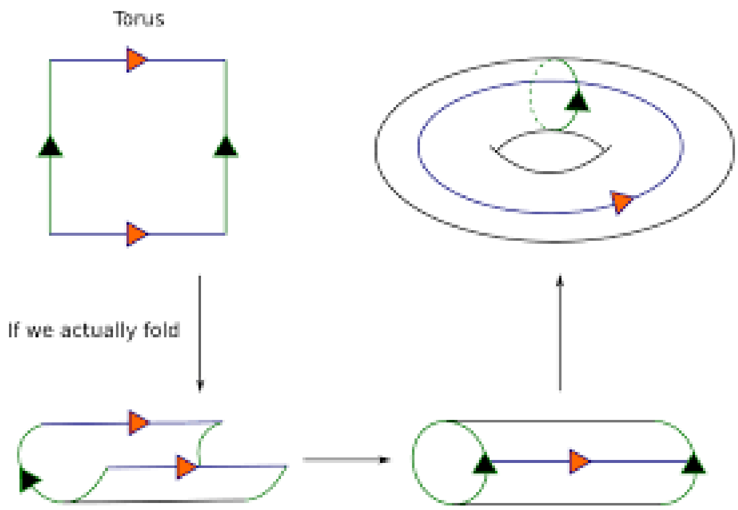

Figure 5.2: Gluing to make a Torus

As in the figure, the two black arrows are going to glued and the two orange arrows are going to be glued together. While gluing we also need to ensure that the direction of arrows of the glued sides is the same as the figure illustrates.
Ex:*Can you come up with such gluing models for double torus?*
Remark:*There can be many models, try to come up with more models!*

CHAPTER 6

Euler characteristic of Surfaces

In this chapter we will use the paper models we constructed to calculate the Euler characteristic.

6.1 Euler characteristic of torus

Let us recall the paper model for a torus first. Let us count the faces first. As

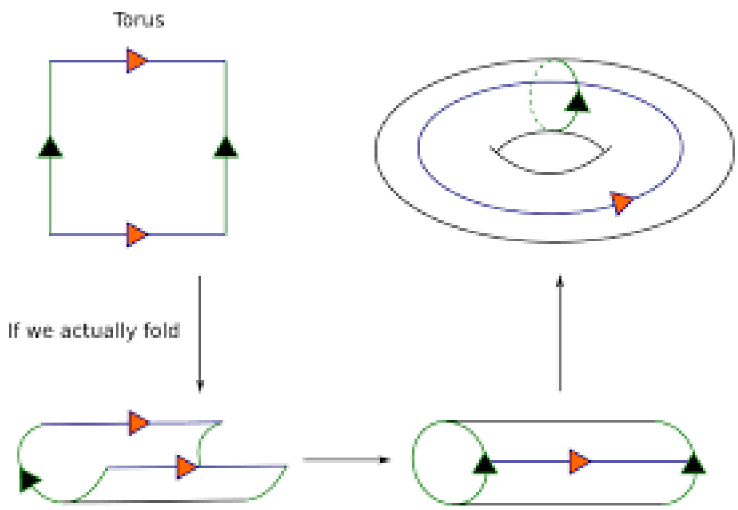

Figure 6.1: Gluing to make a Torus

you can see there is exactly one face for a square. So $F = 1$. Counting edges is slightly tricky. We start out with four edges for a square. But we are identifying the sides. So that cuts the number of edges to just 2 after identification. So $E = 2$. Again for vertices, we start out with four vertices, but due to gluing they all identify to one point. So $V = 1$. This gives us that the Euler characteristic is $\chi = 0$.

6.2 Euler characteristic of a double torus

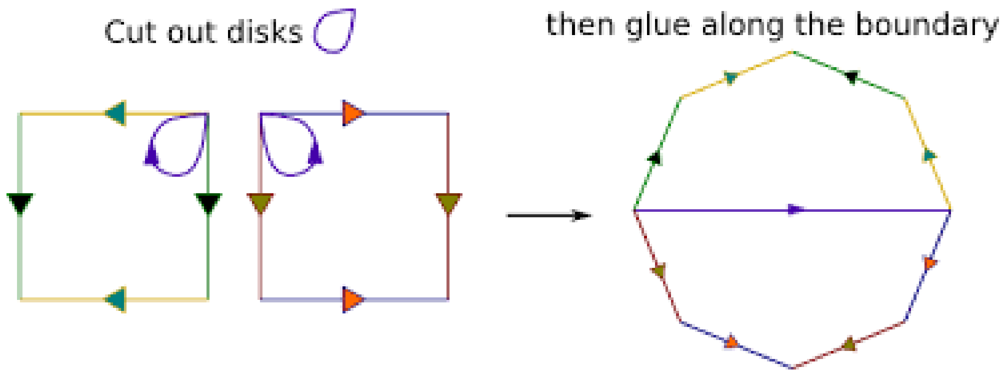

Figure 6.2: Gluing to make a Torus

As picture illustrates, a double torus can be constructed from two torii by cutting out a disc from both of them and gluing along the disc. The line in the centre you see is a circle(Why?). Similarly to the previous model, colors and arrows indicate the which and how to glue edges.

Ex:*Calculate the Euler characteristic of a double torus using this model.*

6.3 Formula for the Euler characteristic

A double torus is what we call a genus 2 surface. The genus essentially counts the number of "holes". So for a genus g surface, the Euler characteristic is given by $2 - 2g$. We can prove this either by constructing explicit models for these genus g surfaces or by trying to observe what happens when you add an extra "hole".

Ex:*Use any of the methods above to convince yourself that the formula is true.*

CHAPTER 7

Loops and homotopy

7.1 Loops

We will start out by what we mean by a loop. A loop is a continuous path which starts and ends at the same point. We will also be fixing this point. For example, you could start from your house, go to school and then come back. That is a loop! Or you could just sit in one place. Even that is a loop. In a loop you can also come back to the beginning many times.

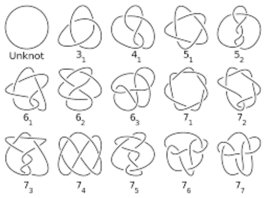

Figure 7.1: Knots

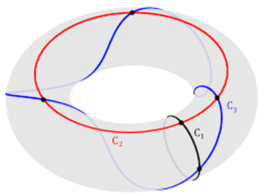

Figure 7.2: Torus loops

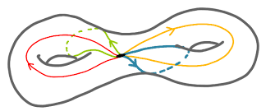

Figure 7.3: Double torus loops

7.2 Homotopy

Since we are interested in topology. We look at objects up to deformation. Similarly we will only care about loops up to a deformation. So if we can deform one to the other they are the same. For example, if you two parallel loops on the torus, they are the same.In the picture below, blue and green are the same loop but the black loop is different.

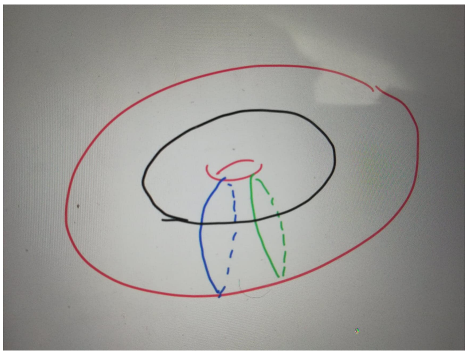

Figure 7.4: Homotopy

7.3 Zero loop

Suppose you can deform a loop to a point. We call such loops zero loops. We will primarily be interested in non zero loops. So these are the loops that encompass a hole inside!

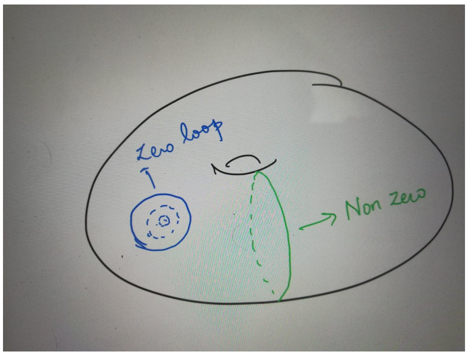

Figure 7.5: Zero loop

7.4 Groups

A group is a bunch of objects which we can add and subtract. For example, if you take the integers, we can add and subtract them. So integers form a group. But if you just take the natural numbers you cannot subtract 2 from 1. So it does not form a group.The reason we are discussing this is because the loops form a group!

7.5 Group structure on Loops

So how do we sdd loops. We just put them together and combine them. For subtracting we just reverse the direction of the loop. These are best understood through pictures!

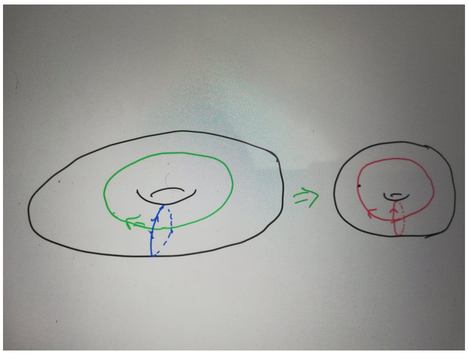

Figure 7.6: Adding loops

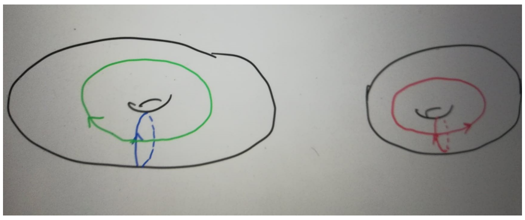

Figure 7.7: Subtracting loops

7.6 Rings

We saw groups before. We said something was a group if you could add and subtract things. For a ring, we will also require that we can multiply things. For example, in integers we can add, subtract and multiply. So integers form a ring. We will see that loops also form a ring.

7.7 Multiplicative structure on Loops

For multiplying loops, we take the number of points of intersections of the two loops. But unfortunately if we change the loops a little bit, this number can change. But miraculously, the parity does not change. So we have two possibilities for multiplication of two loops i.e odd or even.

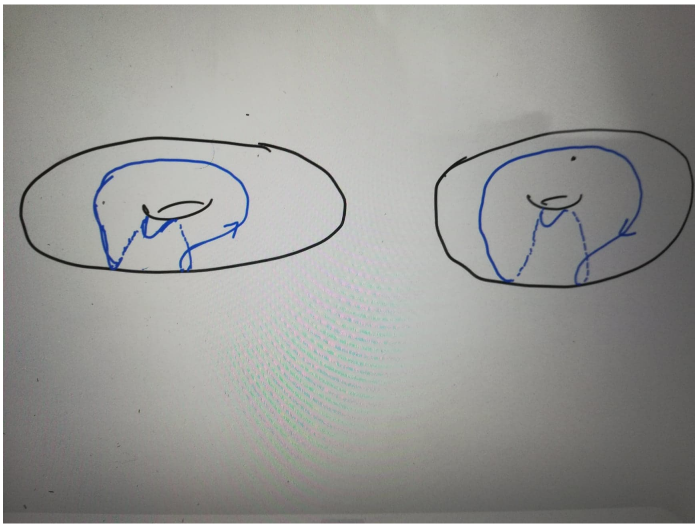

Figure 7.8: Reversing a loop

CHAPTER 8

Geometry

So far we have studied shapes up to deformation. Now we will start being more rigid about what deformations are allowed. Now we will only allow deformations that do not change distances between points. For example, a deformed sphere does not have the same geometry as a nice symmetric sphere even though they have the same topology.

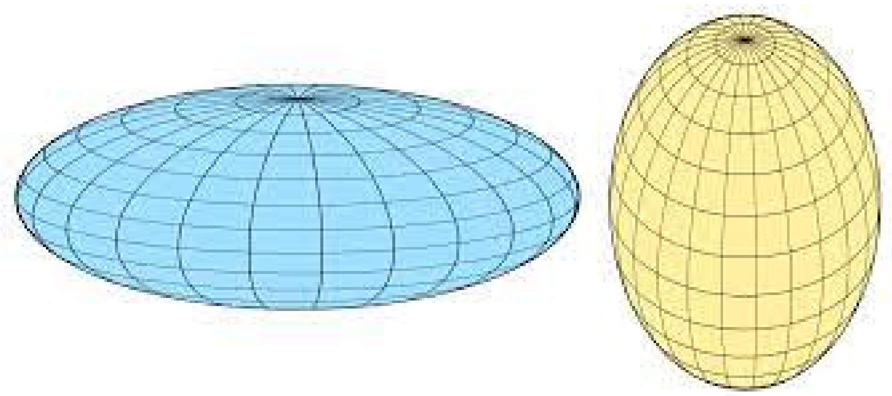

Figure 8.1: Deformed sphere has a different geometry

Some questions to ask given a geometry

1. *We had invariants to study different topologies, what is an invariant to study different geometries?*
2. *What are the straight lines in the geometry?*

Before we answer these questions, let's try to think of some different geometries.

8.1 Euclidean geometry/Flat geometry

This is precisely the geometry on the flat plane. Once I tell you the distance between the points you can ask the question **Given two points P and Q, what is the shortest distance between them?**. The path that realises this shortest distance is called a geodesic/straight line. This geometry satisfies 5 postulates which encapsulate all it's properties.

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough. This postulate is equivalent to what is known as the parallel postulate.

This is a geometry we already understand very well!

8.2 Spherical/Elliptic geometry

This as the name suggests is the geometry on a sphere.

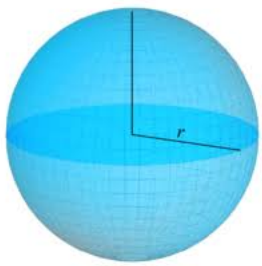

Figure 8.2: Spherical Geometry

Ex: *What are geodesics on a sphere?* This geometry has many "weird properties".
Theorem: *The angles of a triangle are more than 180 degrees.*
Theorem: *There are not parallel lines. All lines intersect in two points.*

8.3 Hyperbolic geometry

This is the geometry of the saddle.

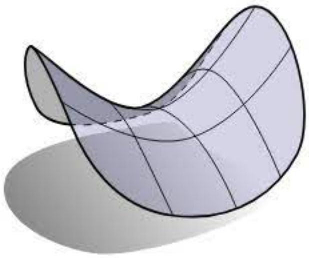

Figure 8.3: Hyperbolic Geometry

But we will use a different model which will help us understand this geometry better. The model we will use is called the Poincare disc model.

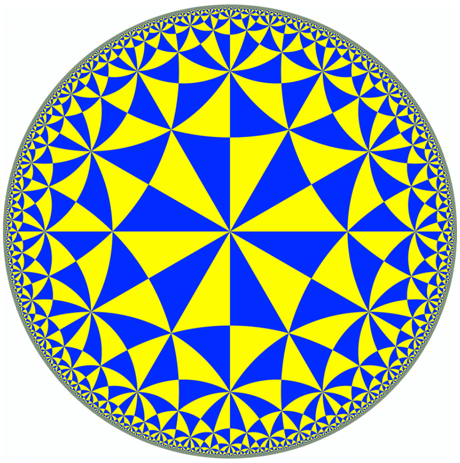

Figure 8.4: Poincare disc model

The triangles you see in the picture above all have the same area. So we need to think of it as a distorted model of hyperbolic geometry. Due to the distortion the distances get very large as you approach the boundary. Therefore the triangles look smaller even though they have the same area.

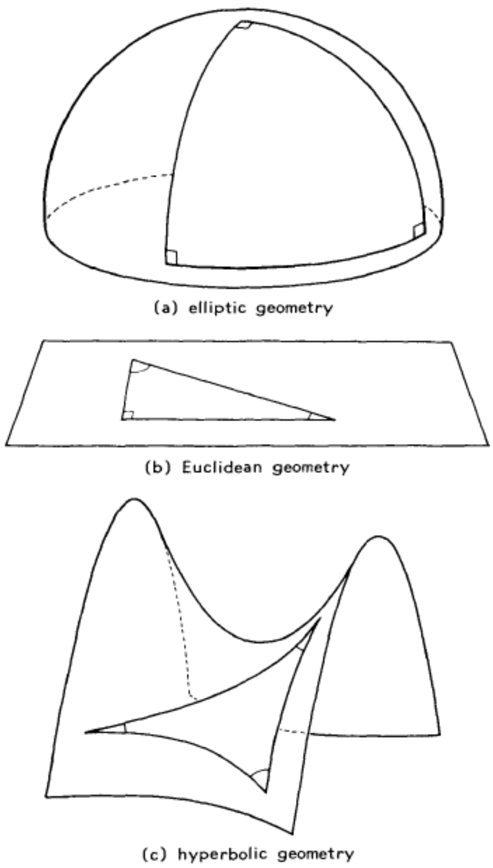

Figure 10.1 The three homogeneous two-dimensional geometries.

Figure 8.5: The three geometries

8.4 What is curvature?

Let us start with a curve. There are many ways to define the curvature of a curve. We will talk about two possible definitions which are equivalent. We say the curvature of a curve to be the radius of the circle which approximates the curve best at the point. This circle is called the osculating circle.

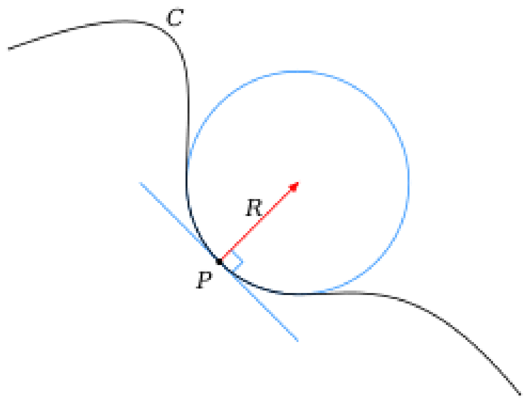

Figure 8.6: Curvature

Another alternate definition uses the observation that, when there is curving, the tangent vector changes. If θ is the angle between the tangent line to the curve at point and the x axis, then $\kappa =$ rate of change of θ, where κ denotes the curvature. Now let us try to see how to define curvature for surfaces.

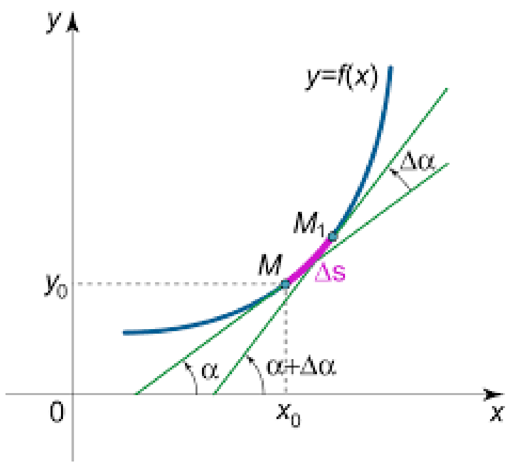

Figure 8.7: Curvature

8.5 Gaussian Curvature

Theorem:*Every point on the surface has two principal directions where the curvature is maximum or minimum. These are called principal curvatures.*
Remark:*Principal curvature is not intrinsic but fortunately their product is intrinsic. So we can define the curvature to the product of maximum and minimum curvature*

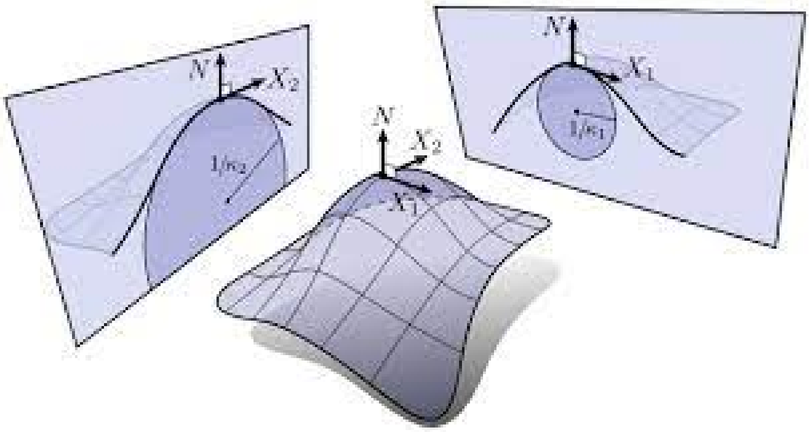

Figure 8.8: Principal Curvatures

The figure below shows the directions of principal curvature on a curved plane and a cone. I want to specifically emphasize the curvature of a cone, on every point apart from the sharp cone point, the gaussian curvature is zero because there is a flat direction always.

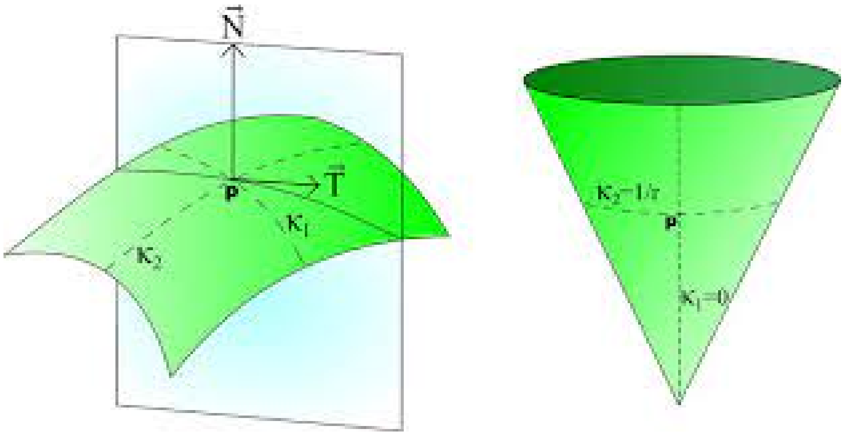

Figure 8.9: Curvatures on Curved plane and Cone

The figure below shows the gaussian curvature on a torus.

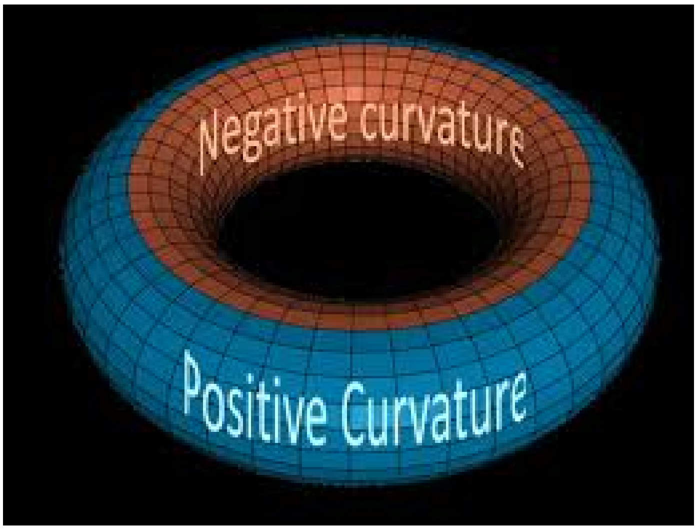

Figure 8.10: Gaussian Curvature on a Torus

CHAPTER 9

Gauss Bonnet and Uniform Geometries

Theorem:*For a surface with curvature κ, we have $\kappa A = 2\pi\chi$. This is remarkable since it is relating κ(geometry) with χ(topology).*

9.1 Uniform Geometry

Definition:*A surface is said to have uniform curvature if the Gaussian curvature at every point is the same.*
Question:*What shapes can have uniform geometries?* We already know a partial answer to this question. For example sphere has a constant curvature.

9.2 Uniform flat geometry on a Torus

We can obtain a flat torus by just using the flat paper and gluing sides. Note

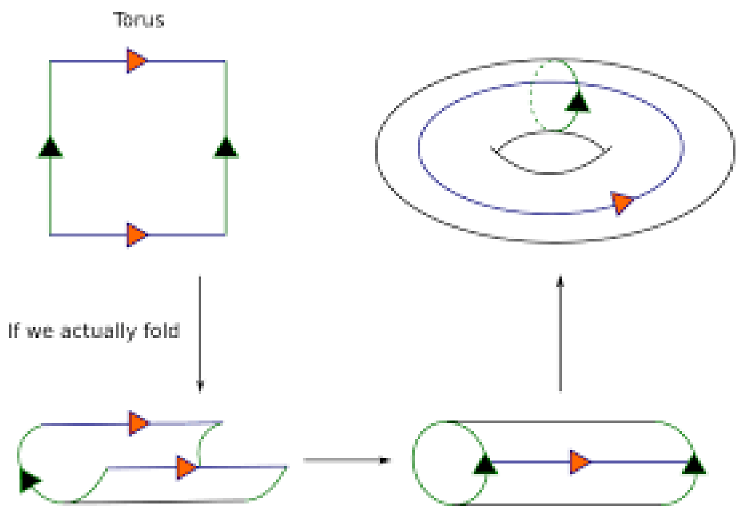

Figure 9.1: Flat torus model

we will not create any sharp edges because the angles at the four vertices add up exactly to a 360.

9.3 Uniform hyperbolic geometry on a double torus

If we take the usual model for the double torus, then the interior angles of an octagon add up to more than 360. So it will create a sharp point when you actually glue it together.
So to solve this issue, instead of taking plane sheets of paper, we will image that

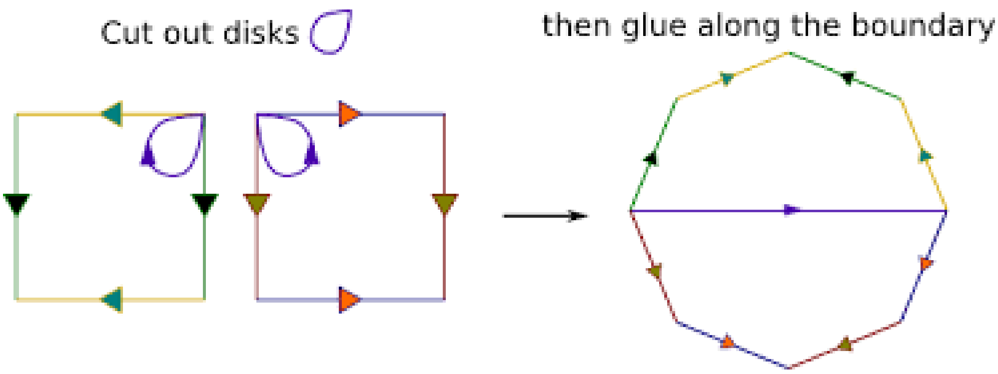

Figure 9.2: Gaussian Curvature on a Torus

we a poincare disc and try to do the exact same construction on the Poincare disc.

In the picture below, observe that due to hyperbolic geometry, the interior angles of the octagon keep decreasing as it grows in size. So the interior angles start out a little less than 135 and become almost 0 when the octagon becomes larger. So at some point the interior angles will be exactly 45. So then we can glue all the angles together without producing a sharp point.
Question:*Can you define a uniform hyperbolic geometry on a triple torus using the same method?*

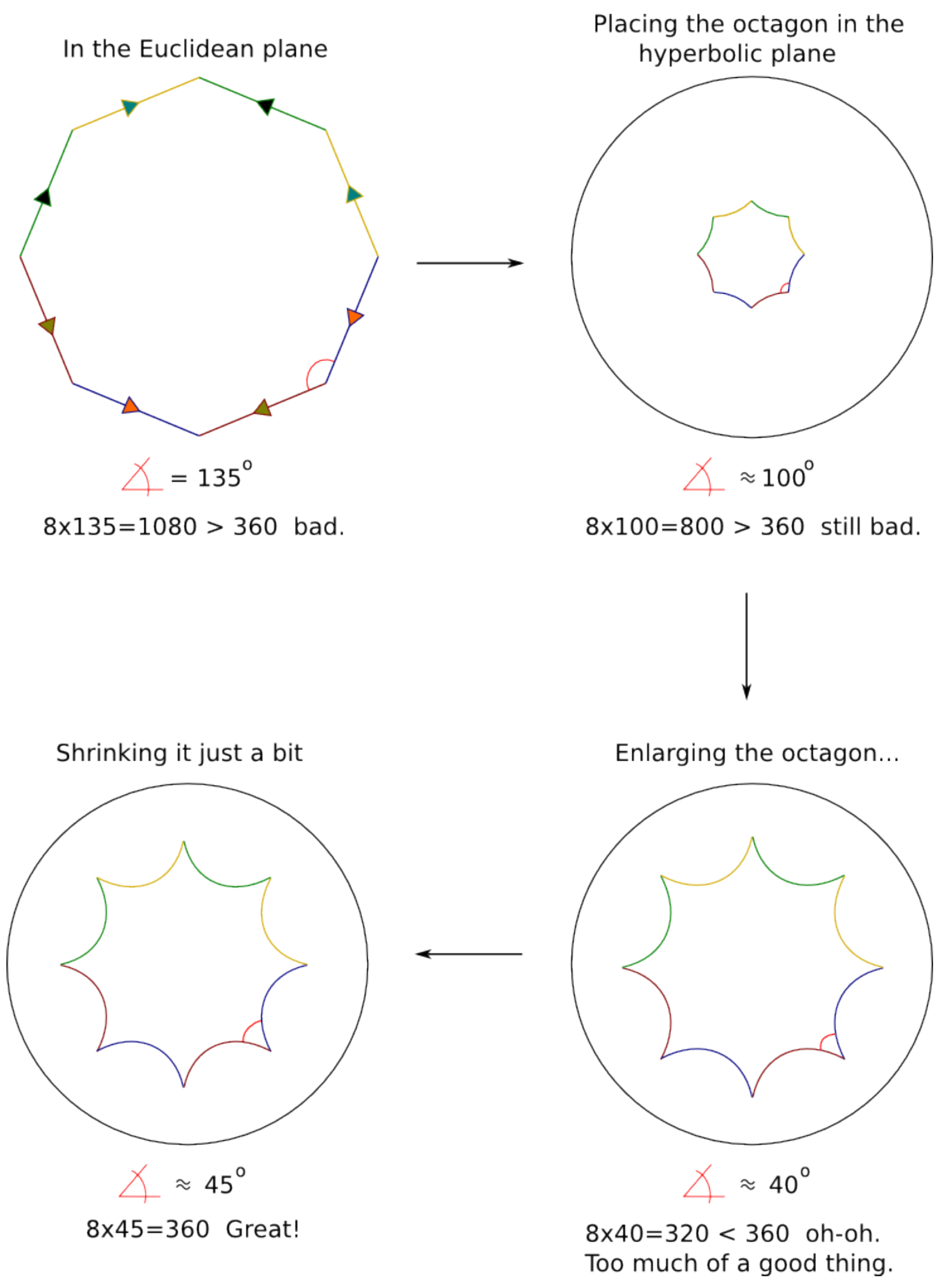

Figure 9.3: Double torus with hyperbolic geometry

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