INVERTIBLE OBJECTS IN STABLE HOMOTOPY THEORY

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Abstract. These are the notes taken by Venkata Sai Narayana Bavisetty and Saad Slaoui - the notetakers are most likely to blame for any mistakes herein.

1. Algebra

In abstract algebra, if R is a commutative ring, we can make the following three constructions:

	Algebraic interpretation	Categorical interpretation	Cohomological interpretation
$R^\times$	Group of units of R	Multiplicative units in R	$H^0(\mathrm{Spec}(R), \mathbb{G}_m)$
$\mathrm{Pic}(R)$	Picard group of invertible R-modules up to equivalence	Units for $\otimes_R$ in Mod_R	$H^1(\mathrm{Spec}(R), \mathbb{G}_m)$
$\mathrm{Br}(R)$	Brauer Group of Azumaya R-algebras up to equivalence	Units for $\otimes_{Mod_R}$ in the category Cat_R of R-linear categories	$H^2(\mathrm{Spec}(R), \mathbb{G}_m)$[1]

Here, $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[t^\pm])$ denotes the multiplicative group, and we are taking étale cohomology groups. Note that all these invariants are abelian groups and that they are obtained by taking units one categorical level higher! We may think of these invariants as capturing arithmetic information about the ring R.

Question:
- *Can we find analogous interpretations of $H^3(SpecR, \mathbb{G}_m)$?*
- *Can we make similar constructions in the world of stable homotopy theory?*

In this talk, we focus on the latter question.

2. Stable homotopy theory

In the following, R will denote a commutative ring spectrum, i.e. a commutative algebra object in the ∞-category of spectra.

2.1. **Units of R (70s, [MQRT]).** The space of units is denoted by $GL_1(R)$ and defined via the following pullback diagram:

$$\begin{tikzcd} GL_1(R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Omega^\infty(R) \arrow[d] \\ \pi_0(R)^\times \arrow[r] & \pi_0(R) \end{tikzcd}$$

One can check that $GL_1(R)$ is a group-like infinite loop space, so that we can associate to it a spectrum of units $gl_1(R)$, whose infinite loop space recovers $GL_1(R)$. However, the inclusion of spaces $GL_1(R) \to \Omega^\infty R$ is not even a based map. Even if we make it based by subtracting

[1]This is not quite right for a non-regular ring R.

1, it is still not a map of $\mathbb{E}_\infty$-spaces. Therefore, we do not have a natural map of spectra from $gl_1(R)$ to R.

The corresponding statement in algebra is the fact that the inclusion $R^\times \hookrightarrow R$ does not preserve the group structure. Here, the group structure on $R^\times$ is given by multiplication, whereas the group structure on R is given by addition.

Exercise 1: *Compute the homotopy groups* $\pi_*(GL_1(R))$

Exercise 2: *Show that, for any space* X*, we can identify* $[X, Gl_1(R)] \cong R^0(X)^\times$.

Vista: **[Rezk]** *"The units of a ring spectrum and a logarithmic cohomology operation".*
There is a $K(n)$-local logarithm
$$gl_1(R) \xrightarrow{log} L_{K(n)}R,$$
where $K(n)$ denotes Morava K-theory of height n.

In algebra we can identify $R^\times = \mathrm{Map}_{\mathrm{Rings}}(\mathbb{Z}[t^\pm], R)$. Similarly, in stable homotopy theory, we have that:
$$GL_1(R) \simeq \mathrm{Map}_{\mathrm{CAlg}}\Big(\text{Free comm. alg. on one invertible generator}, R\Big).$$

2.2. **Strict units of** R**.** The strict units of R, denoted by $\mathbb{G}_m(R)$, are defined as follows:
$$\mathbb{G}_m(R) := \mathrm{Map}_{\mathrm{CAlg}}(S^0[t^\pm], R) \simeq \mathrm{Map}_{\mathrm{Sp}}(H\mathbb{Z}, gl_1(R)).$$

Exercise 3: *Try to compute the homotopy groups* $\pi_*(\mathbb{G}_m(R)$*, and see how hard to impossible it is.*

2.3. **The Picard Space/Spectrum [ABGHR1].** The Picard space, denoted by $\mathcal{P}ic(R)$, is defined as
$$\mathcal{P}ic(R) := \big\{\text{space of invertible } R\text{-modules and equivalences between them}\big\}.$$
It is a group like $\mathbb{E}_\infty$-space under tensor product $\otimes_R$.

Note: $\pi_0(\mathcal{P}ic(R)) = \big\{\text{invertible } R\text{-modules modulo equivalences}\big\}$ retrieves the classical Picard group of *graded* modules.

Some properties of $\mathcal{P}ic(R)$:
(1) $\Omega\mathcal{P}ic(R) = Gl_1(R)$
(2) $(\mathcal{P}ic(R))_R = BGL_1(R)$, where $(\mathcal{P}ic(R))_R$ denotes the connected component containing R.
(3) $[X, \mathcal{P}ic(R)] =$ Bundles of invertible R-modules on X.
(4) $[X, \mathcal{P}ic(S^0)] =$ Stable spherical fibrations on X.

Since $\mathcal{P}ic(R)$ is an $\mathbb{E}_\infty$-space, we get an associated connective spectrum $pic(R)$.

Exercise 4: *Classify invertible* S^0 *modules, i.e. compute* $\pi_0(\mathcal{P}ic(S^0))$.

Example: *Let ko be the connective* K*-theory spectrum. Then* $\Omega^\infty ko \simeq \mathbb{Z} \times BO$ *classifies stable real vector bundles. Given a vector bundle* $\xi \to X$*, we can take its Thom spectrum*

$Th(\xi)$, which is a spherical fibration. Therefore, we get a map of the classifying objects i.e we get a map of $\mathbb{E}_\infty$-spaces

$$\mathbb{Z} \times BO \to \mathcal{P}ic(S^0),$$

which gives rise to a map of spectra

$$ko \to pic(S^0).$$

This map is precisely Adams' J-homomorphism.

Why did people study all these?

Vista: **[[MQRT], [ABGHR2]]** *"Orientations and Thom Spectra"*
To a map $X \xrightarrow{f} BGL_1(R)$, we can associate a Thom spectrum:

$$Mf := \operatorname{colim}(X \xrightarrow{f} BGL_1(R) \hookrightarrow Mod_R).$$

Vista: **[AS], [FHT], [ABG], [SW].**

The ideas of this section are involved in the study of twisted R-cohomology, as well as twists by ordinary cohomology classes, e.g. via strict $\mathbb{P}ic = \operatorname{map}(H\mathbb{Z}, pic)$.

2.4. **Brauer Space.** This is much harder to work with. One may consult **[BRS], [AG], [GL], . . .**

3. Computations

Well, how do we compute anything? For our purposes we will assume that the homotopy groups of R are known and go from there.

We may look at:

(1) $\pi_0(\mathcal{B}r(R))$

(2) $\pi_1(\mathcal{B}r(R)) = \pi_0(\mathcal{P}ic(R))$

(3) $\pi_2(\mathcal{B}r(R)) = \pi_1(\mathcal{P}ic(R)) = \pi_0(GL_1(R)) = \pi_0(R)^\times$.

(4) $\pi_{>2}(\mathcal{B}r(R)) = \pi_{>1}(\mathcal{P}ic(R)) = \pi_{>0}(GL_1(R)) = \pi_{>0}(R)$.

We are also interested in the k-invariants.

Example: **First k-invariant of Pic(R):**

$$pic(R) \to H(\pi_0(pic(R))) \xrightarrow{k} \Sigma^2 H(\pi_1(pic(R)))$$

So the k-invariant is equivalent to the data of a homomorphism:

$$\pi_0(pic(R)) \to \pi_1(pic(R))[2] = \pi_0(R)^\times[2].$$

The fact that this k-invariant has to be 2 torsion follows from its explicit description. Any $L \in \pi_0(pic(R))$ comes with a twist map

$$\tau : L \otimes_R L \xrightarrow{\simeq} L \otimes_R L.$$

Since $L \otimes_R L \in \pi_0(pic(R))$, we get an element of $\pi_0(R)^\times[2]$. This map precisely is the k-invariant and it is clearly 2-torsion.

4. Approaches

We will see three approaches to understanding these structures:

(1) Comparison with algebra ("reduction to an easier problem").
(2) Descent.
(3) Obstruction theory.

4.1. **Comparison with algebra.** We have an injective map

$$0 \to Pic(\pi_*(R)) \xrightarrow{i} \pi_0(pic(R)),$$

where $\mathrm{Pic}(\pi_*(R))$ denotes the graded Picard group of $\pi_*(R)$. This map is constructed by using the fact that $N \in Pic(\pi_*(R))$ is automatically flat over $\pi_*(R)$. We can use a presentation of N to build this map.

Theorem:**[BR]** *The map i is an isomorphism in either of the following cases*

(1) *R is connective.*
(2) *R is weakly even periodic and $\pi_0(R)$ is a regular Noetherian ring.*

By weakly even periodic, we mean that $\pi_{\mathrm{odd}} R = 0, \pi_2(R)$ is an invertible $\pi_0(R)$-module, and $\pi_{2k} R \simeq \pi_2 R^{\otimes k}$ for every $k \geq 0$.

Vista:*The analogue for $Br(R)$ is much harder. See* [**BRS**], [**GL**].

4.2. **Descent. Theorem**:**[ABGHR3], [AG], [GL], [MS], [AMS]**
The functors

$$\mathcal{P}ic, \mathcal{B}r : \mathrm{CALg(Sp)} \to \big(\text{infinite loop spaces}\big)$$

both satisfy étale and Galois descent.

Here, we use Galois descent in the sense of Rognes. Note that in the settings of ring spectra, étale descent does **NOT** imply Galois descent. A map of ring spectra $R \to S$ is said to be étale if

(1) The induced map $\pi_0(R) \to \pi_0(S)$ is étale as a map of commutative rings, and
(2) $\pi_*(S) \xleftarrow{\simeq} \pi_*(R) \otimes_{\pi_0(R)} \pi_0(S)$ for all $k \geq 1$.

Next, if S is a Borel G-spectrum, we say that $R \to S$ is a G-Galois extension if

(1) The induced map $R \to S^{hG}$ is an isomorphism.
(2) We have an equivalence $S \otimes_R S \simeq \Pi_G S$.

Here, S^{hG} denotes the homotopy fixed points of S under the G-action.

Example: KO has no interesting étale extensions, but it has an interesting C_2 Galois extension. Namely, the map $KO \to KU$ is a C_2 Galois extension.

Expanding upon the above example, we have that $\pi_*(KU) = \mathbb{Z}[\beta^{\pm}]$ and $\mathrm{Pic}(\pi_*(KU)) = \mathbb{Z}/2\mathbb{Z}$. This implies that $\mathrm{Pic}(KU) = \mathbb{Z}/2\mathbb{Z}$.
So, by descent, we have that $\mathcal{P}ic(KO) = \mathcal{P}ic(KU)^{hC_2}$. The right hand side is more computable, as we have access to a homotopy fixed point spectral sequence. See [**GL**], [**MJ**] for more details.

Applications: Suppose $R \to S$ is a G-Galois extension, where S is an even periodic G-spectrum. Then this framework allows us to calculate $\pi_0(pic(R))$ and the relative Brauer group associated to this situation - even though the absolute Brauer space $\mathcal{B}r(S)$ itself may be more mysterious. There are also applications relating to étale locally trivial Brauer classes.

4.3. **Obstruction theory.** See **[HL]**. This approach is more difficult and unfortunately we did not have time to cover it.

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