DESCENT AND PICARD GROUP OF $Q(2)$

BY

VENKATA SAI NARAYANA BAVISETTY

DISSERTATION

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Doctoral Committee:

Professor Charles Rezk, Chair
Professor Vesna Stojanoska, Director of Research
Assistant Professor Daniel Berwick Evans
Professor Randy McCarthy

Abstract

For l a topological generator of the p-adic units $\mathbb{Z}_p^\times$, where p is any prime, Behrens [8] introduced a semi cosimplicial spectrum $Q(l)$ which has a resolution constructed using topological modular forms, TMF, and related spectra. When p is 3 and l is 2, the spectrum $Q(2)$ is closely related to the $K(2)$-local sphere.

In my thesis, I investigate the Picard group of $Q(l)$. I prove some descent and detection results for invertible $Q(l)$-modules. For computations, I restrict l to 2 at the prime 3, and using descent I calculate some elements in the Picard group of $Q(2)$. I also prove detection results for the elements of Picard group of the $K(2)$-local category of spectra.

To my parents.

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Chapter 1

Introduction

Picard groups were first studied in number theory as class groups of number fields. Later this notion was generalised to define the Picard group for a scheme X. Abstractly, one can define the Picard groups for any symmetric monoidal category by taking the group of isomorphism classes of objects invertible under the tensor product. Symmetric monoidal categories abound in homotopy theory, and some natural examples are the category of modules over an $\mathbb{E}_\infty$-ring R and the $K(n)$-local category of spectra, where $K(n)$ is the Morava K-theory spectrum at height n. In homotopy theory, interest in Picard groups arose once Hopkins, Mahowald and Sadofsky[20] noted that the Picard group of the $K(n)$-local category of spectra is non-trivial and may contain *exotic elements*.

For l a topological generator of the p-adic units $\mathbb{Z}_p^\times$, where p is any prime, Behrens [8] introduced a semi cosimplicial spectrum $Q(l)$ which has a resolution constructed using topological modular forms, TMF, and related spectra. When p is 3 and l is 2, the spectrum $Q(2)$ is closely related to the $K(2)$-local sphere, $L_{K(2)}S^0$ and in fact satisfies a cofiber sequence [8]

$$DQ(2) \to L_{K(2)}S^0 \to Q(2), \tag{1.1}$$

where $DQ(2)$ is the $K(2)$-local Spanier-Whitehead dual of $Q(2)$.

In this thesis, we investigate the Picard group of $Q(l)$. We prove some descent and detection results for $\mathfrak{pic}(Q(l))$. For computations, we restrict l to 2 at the prime 3, and using descent we calculate some elements in the Picard group of $Q(2)$. We also prove detection results for the elements of Picard group of the $K(2)$-local category of spectra.

1.1 Picard spectra

We begin by recalling the definition of a Picard spectrum for a presentable symmetric monoidal ∞-category from [31].

Definition 2.12 ([31, Definition 2.2.1]). *Given a presentable symmetric monoidal ∞-category $(\mathcal{C}, \otimes, \mathbb{1})$, let $\mathcal{P}ic(\mathcal{C})$ denote the ∞-groupoid of invertible objects in C and equivalences between them. $\mathcal{P}ic(\mathcal{C})$ is a group-like $\mathbb{E}_\infty$-space, and thus the delooping of a connective Picard spectrum $\mathfrak{pic}(\mathcal{C})$[32].*

The Picard spectrum is very well behaved homotopically. For example, the functor $\mathfrak{pic}$ commutes with limits and filtered colimits[31, Proposition 2.2.3]. The 0-th homotopy group of $\mathfrak{pic}(\mathcal{C})$ is the Picard group $\mathrm{Pic}(\mathcal{C})$.

Descent for the Picard spectrum at height 1

At an odd prime p, we have a finite resolution for the $K(1)$-local sphere

$$L_{K(1)}S \longrightarrow K \underset{\mathbb{1}_K}{\overset{\psi_\gamma}{\rightrightarrows}} K\ , \tag{1.2}$$

where γ is a topological generator of $\mathbb{Z}_p^\times$, K is the p-completed complex K-theory, ψ_γ is the Adams operation and $\mathbb{1}_K$ is the identity. Since the $K(1)$-local sphere is the limit of a diagram in the category of $\mathbb{E}_\infty$-rings we get a functor

$$\mathfrak{A} : Sp_{K(1)} \to \lim\left(\mathrm{Mod}(K) \underset{\mathbb{1}_K^*}{\overset{\psi_\gamma^*}{\rightrightarrows}} \mathrm{Mod}(K)\right), \tag{1.3}$$

where $\mathrm{Mod}(K)$ is the category of $K(1)$-local K-modules, and ψ_γ^* and $\mathbb{1}_K^*$ are the functors obtained by tensoring up along the maps ψ_γ and $\mathbb{1}_K$ respectively. Applying $\mathfrak{pic}$ and using the fact that $\mathfrak{pic}$ commutes with limits, we get a functor

$$\mathfrak{A}|_{\mathfrak{pic}} : \mathfrak{pic}(Sp_{K(1)}) \to \lim\left(\mathfrak{pic}(K) \underset{\mathbb{1}_K^*}{\overset{\psi_\gamma^*}{\rightrightarrows}} \mathfrak{pic}(K)\right). \tag{1.4}$$

Using the description of the right adjoint of $\mathfrak{A}$ outlined in Section 3.2, we obtain the following rephrasing of [20, Proposition 2.1].

Theorem 4.9. *The functor $\mathfrak{A}|_{\mathfrak{pic}}$ in (1.4) is an equivalence.*

This descent theoretic interpretation of [20, Proposition 2.1] has the advantage that it constructs the invertible elements in a purely formal way by simply taking the right adjoint of $\mathfrak{A}$ (see Chapters 3 and 4).

The spectrum $Q(l)$

In the following, we assume that all spectra are $K(2)$-local. The Goerss-Hopkins-Miller theorem produces a sheaf of E_∞-ring spectra $\mathcal{O}^{top}_{GHM}$ on the small étale site of $\mathcal{M}_{ell}$, where $\mathcal{M}_{ell}$ is the moduli stack of elliptic curves. Further, by [10], the sheaf $\mathcal{O}^{top}_{GHM}$ can be extended to a sheaf $\mathcal{O}^{top}$ which is functorial with respect to isogenies of elliptic curves of invertible degree.

The spectrum $Q(l)$ is the global sections $\Gamma\left(\mathcal{M}_\bullet, \mathcal{O}^{top}\right)$ of a derived semi simplicial stack $\mathcal{M}_\bullet$, defined as

$$\mathcal{M}_\bullet \coloneqq \lim\left(\mathcal{M}_{ell} \leftleftarrows \mathcal{M}_0(l) \amalg \mathcal{M}_{ell} \leftleftarrows\!\!\!\!\leftarrow \mathcal{M}_0(l)\right), \tag{1.5}$$

where $\mathcal{M}_0(l)$ is the moduli stack of elliptic curves with $\Gamma_0(l)$ level structures. The global sections of $\mathcal{M}_{ell}$ and $\mathcal{M}_0(l)$ are called TMF and $TMF_0(l)$ respectively. Since, all the maps in (1.5) arise from isogenies of elliptic curves (see Section 2.4) we can take global sections of $\mathcal{O}^{top}$ to get a semi cosimplicial diagram

$$Q(l) \coloneqq \lim\left(TMF \rightrightarrows TMF_0(l) \times TMF \rightrightarrows\!\!\!\!\rightarrow TMF_0(l)\right). \tag{1.6}$$

Working with $Q(l)$ instead of $L_{K(2)}S^0$ has many advantages. Firstly, the maps in the resolution of $Q(l)$ arise from maps of the moduli stack of elliptic curves, which allows the use of number theoretic techniques.

For example, one of the maps in the resolution of $Q(l)$ is given by

$$\begin{aligned}\psi_d :&\mathcal{M}_0(l) \to \mathcal{M}_0(l)\\ &(C,H) \to (C/H, Ker(\hat{\phi}_H)),\end{aligned} \tag{1.7}$$

where (C,H) is an elliptic curve with level structure and $\hat{\phi}_H$ is the dual isogeny corresponding to the map $\phi_H : C \to C/H$. The map induced by ψ_d on modular forms is classically known as the Atkin-Lehner involution and is amenable to study via the theory of modular forms. Secondly, $Q(l)$ is the limit of a diagram in the category of $\mathbb{E}_\infty$-rings. This allows the use of descent theoretic techniques to understand the category of $K(2)$-local modules over $Q(l)$, $\mathrm{Mod}(Q(l))$, in terms of the category of $K(2)$-local modules over TMF and $TMF_0(l)$ denoted as $\mathrm{Mod}(TMF)$ and $\mathrm{Mod}(TMF_0(l))$ respectively. In particular, we have a functor

$$F : \mathrm{Mod}(Q(l)) \to \lim\left(\mathrm{Mod}(TMF) \rightrightarrows \mathrm{Mod}(TMF_0(l)) \times \mathrm{Mod}(TMF) \substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow} \mathrm{Mod}(TMF_0(l)) \right), \tag{1.8}$$

and we denote by $\mathcal{L}$ the limit category which is the target of F in (1.8). Using [26, Theorem 7.2], we get that F is a fully faithful functor. Applying $\mathfrak{pic}$, we get a functor

$$F|_{\mathfrak{pic}} : \mathfrak{pic}(Q(l)) \to \mathfrak{pic}(\mathcal{L}). \tag{1.9}$$

1.2 Summary and results

The goal of my thesis is to construct invertible $Q(l)$-modules using elements of $\mathrm{Pic}(\mathcal{L})$. The strategy we employ can be summarized as follows. Firstly, we calculate an explicit formula for the right adjoint G of F in Theorem 3.3. Secondly, we prove a descent result in Theorem 4.10. Using Theorem 4.10, we prove a detection theorem for when $G(X) \in \mathrm{Pic}(Q(l))$ in Theorem 4.11. Finally in Theorem 5.1, we use these results to compute $\mathrm{Pic}(\mathcal{L})$ in the case l is 2 at the prime 3. In Theorem 5.8 we use invertible elements in the $K(2)$-local category to construct some non-trivial elements in $\mathrm{Pic}(Q(2))$.

Given an object in $\mathrm{Pic}(\mathcal{L})$ we would like to construct an object in $\mathrm{Pic}(Q(l))$. Since F preserves small colimits, the adjoint functor theorem gives a right adjoint G, which allows us to produce potential elements in $\mathrm{Pic}(Q(l))$. In what follows we describe an explicit construction of the right adjoint G.

An object in the limit category $\mathcal{L}$ is a *compatible system* of objects (M_0, M_1, M_2) where $M_0 \in \mathrm{Mod}(TMF)$, $M_1 \in \mathrm{Mod}(TMF_0(l)) \times \mathrm{Mod}(TMF)$ and $M_2 \in \mathrm{Mod}(TMF_0(l))$ with various isomorphism data. Following [21, Theorem 5.5], in Theorem 1.1 we construct a semi-cosimplicial diagram whose limit is the right adjoint G. Given an isomorphism class of objects in the limit category, we pick a convenient representative of the form (M_0, ϕ), where M_0 is a TMF-module and $\phi : d_0^* M_0 \to d_1^* M_1$ is an isomorphism satisfying a *cocycle condition.*

Theorem 1.1. *With (M_0, ϕ) as above we have the following, the right adjoint G is given as the limit*

$$G(M_0,\phi) \coloneqq \lim \left(\begin{tikzcd} M_0 \arrow[r, shift left, "\phi\circ(d_0\hat{\otimes}\mathbb{1}_{M_0})"] \arrow[r, shift right, "d_1\hat{\otimes}\mathbb{1}_{M_0}"'] & d_1^* M_0 \arrow[r, shift left=2, "d_2^*(\phi)\circ(d_0\hat{\otimes}\mathbb{1}_{d_1^*M_0})"] \arrow[r, "d_1\hat{\otimes}\mathbb{1}_{d_1^*M_0}" description] \arrow[r, shift right=2, "d_2\hat{\otimes}\mathbb{1}_{d_1^*M_0}"'] & d_2^* d_1^* M_0 \end{tikzcd} \right).$$

In general, the right adjoint of a symmetric monoidal functor is only lax monoidal. Therefore, for an object $X \in \mathrm{Pic}(\mathcal{L}))$, it is not necessary that $G(X) \in \mathrm{Pic}(Q(l))$. Thus, we need a detection theorem for when $G(X) \in \mathrm{Pic}(Q(l))$, as outlined in Theorem 4.11.

Theorem 4.11. *If $X \in \mathrm{Mod}(Q(l))$, then the following are equivalent.*

1. *X is an invertible $Q(l)$-module.*
2. *$X \hat{\otimes}_{Q(l)} E_2 \simeq \Sigma^k E_2$ for some integer k.*
3. *$X \hat{\otimes}_{Q(l)} K(2) \simeq \Sigma^k K(2)$ for some integer k.*

To prove this theorem, we follow the strategy in [20] and first prove a descent result for $Q(l)$ in Theorem 4.10. Below we describe the framework for the descent result.

We have a map $f : Q(l) \to E_2$ by identifying $Q(l)$ as the Γ_{Gal} homotopy fixed-points of E_2 [7]. Using the fact that $Q(l)$ is in the thick subcategory generated by TMF and $TMF_0(l)$ we prove the following descent result for the $Q(l)$-modules.

Theorem 4.10. *The map $f : Q(l) \to E_2$ admits descent in the sense of [30, Definition 3.18] and*

$$\mathrm{Mod}(Q(l)) \simeq \lim \Big(\mathrm{Mod}(E_2) \rightrightarrows \mathrm{Mod}(E_2 \hat{\otimes}_{Q(l)} E_2) \Rrightarrow \mathrm{Mod}(E_2 \hat{\otimes}_{Q(l)} E_2 \hat{\otimes}_{Q(l)} E_2) \cdots \Big),$$

where $\mathrm{Mod}(Q(l))$ denotes $Q(l)$-modules in the category of $K(2)$-local spectra, $\mathrm{Mod}(E_2)$ denotes E_2-modules in the category of $K(2)$-local spectra, $\mathrm{Mod}(E_2 \hat{\otimes}_{Q(l)} E_2)$ denotes $E_2 \hat{\otimes}_{Q(l)} E_2$-modules in the category of $K(2)$-local spectra and so on.

For the rest of this section, we use the theoretical results above to construct invertible $Q(2)$-modules at the prime 3. Many of the techniques for proving Theorem 5.1 were inspired by computations for the analogous additive spectral sequence done as part of an undergraduate research project. As a part of the project, Credi and Appleton[1] proved various 3-divisibility results for modular forms which are intricately related to the 3-torsion computations in $Q(2)$.

The Picard group $\mathrm{Pic}(\mathcal{L})$ is given by the 0-th homotopy group of $\mathfrak{pic}(\mathcal{L})$, $\pi_0(\mathfrak{pic}(\mathcal{L}))$. The group $\pi_0(\mathfrak{pic}(\mathcal{L}))$ is computable via a Bousfield-Kan Spectral Sequence with E_1-page given by

$$E_1^{s,t} = \begin{cases} \pi_t(\mathfrak{pic}(TMF)), & \text{for } s = 0 \\ \pi_t(\mathfrak{pic}(TMF_0(2))) \oplus \pi_t(\mathfrak{pic}(TMF)), & \text{for } s = 1 \\ \pi_t(\mathfrak{pic}(TMF_0(2))), & \text{for } s = 2 \\ 0 & \text{for } s > 2. \end{cases} \tag{1.10}$$

Due to the vanishing line above $s = 2$, the spectral sequence degenerates at page 3 and $E_3^{s,t} = E_\infty^{s,t}$. Since, we are only interested in computing $\pi_0(\mathfrak{pic}(\mathcal{L}))$, we can restrict to computations around $t - s = 0$ in the E_1-page. Therefore, the main relevant terms are $\pi_1(\mathfrak{pic}(TMF)) \simeq \pi_0(TMF)^\times$ and $\pi_1(\mathfrak{pic}(TMF_0(2))) \simeq \pi_0(TMF_0(2))^\times$. We now describe the group $\mathrm{Pic}(\mathcal{L})$.

Theorem 5.1. *The natural map $\mathrm{Pic}(\mathcal{L}) \to \mathrm{Pic}(TMF)$ is surjective and we get a short exact sequence*

$$0 \to E_2^{1,1} \to \mathrm{Pic}(\mathcal{L}) \to \mathrm{Pic}(TMF) \to 0, \tag{5.11}$$

[1] Garett Credi and Casey Appleton are currently undergraduate math majors at University of Illinois at Urbana-Champaign. This work was done as a part of the IGL project: Modular forms and homotopy of $Q(2)$.

where $\mathrm{Pic}(TMF) \simeq \mathbb{Z}/72$ *is generated by* ΣTMF*. The kernel* $E_2^{1,1}$ *is a quotient of the subgroup of* $\pi_0(TMF_0(2))^\times \times \pi_0(TMF)^\times$ *consisting of elements of the form* $\{(b,a) : \psi_d(b) \cdot b = a\}$*, where* ψ_d *is the Atkin-Lehner involution defined in* (2.23)*,* $b \in \pi_0(TMF_0(2))^\times$ *and* $a \in 1 + 3\mathbb{Z}_3 \subset \mathbb{Z}_3^\times \subset \pi_0(TMF)^\times$*. Furthermore, as a group,* $E_2^{1,1} \simeq \mathbb{Z}_3^\times \oplus \left(\frac{\pi_0(TMF_0(2))^\times}{Im(\psi_d \cdot id)\pi_0(TMF)^\times}\right)$*, where* $Im\,(\psi_d \cdot id)$ *is the image of the map defined in* (5.27)*.*

The proof of this theorem relies on the explicit description of the action of the Atkin-Lehner involution ψ_d on modular forms. The main complication which arises in this computation is the fact that the rings, $\pi_0(TMF_0(2))$ and $\pi_0(TMF)$ are complete local rings.

By Theorem 5.1, we can represent an element of $\mathrm{Pic}(\mathcal{L})$ as an equivalence class $[(n,b,a)]$. Here, n is an integer, $b \in \pi_0(TMF_0(2))^\times$, and $a \in 1 + 3\mathbb{Z}_3 \subset \mathbb{Z}_3^\times \subset \pi_0(TMF)^\times$ such that $\psi_d(b) \cdot b = a$. As a corollary of Theorem 3.3, we get the formula for the adjoint G, when restricted to $\mathrm{Pic}(\mathcal{L})$.

Corollary 5.5. *The elements in* $\mathrm{Pic}(Q(2))$ *are of the form* $G(n,b,a)$*, for* n *an integer* $\pmod{72}$*,* $b \in \pi_0(TMF_0(2))^\times$*,* $a \in 1 + 3\mathbb{Z}_3 \subset \mathbb{Z}_3^\times \subset \pi_0(TMF)^\times$ *such that* $\psi_d(b) \cdot b = a$ *and* $G(n,b,a)$ *is the limit*

$$G(n,b,a) = S^n \otimes \lim \left(\begin{tikzcd} TMF \arrow[r, shift left, "(b,a)\cdot d_0"] \arrow[r, shift right, "d_1"'] & TMF_0(2) \times TMF \arrow[r, shift left=2, "b\cdot d_0"] \arrow[r, "d_1" description] \arrow[r, shift right=2, "d_2"'] & TMF_0(2) \end{tikzcd} \right)$$

Using Corollary 5.5 we compute the images of the elements in the Picard group of the $K(2)$-local category of spectra, Pic_2 . We have a map $\mathrm{Pic}(\eta) : \mathrm{Pic}_2 \to \mathrm{Pic}(Q(2))$ which is induced by the unit map of $Q(2)$. We know that the algebraic part of Pic_2 is generated by $S^0, S^0\langle det\rangle$ and the exotic part is generated by P and Q [16, 23]. Using the construction of the spectra P and $S^0\langle det\rangle$ in [16] and [3] respectively, we calculate the images of various elements in Pic_2 in Theorem 5.8.

Theorem 5.8. *The spectra* $S^0\langle det\rangle$ *and* P *are detected in* $\mathrm{Pic}(\mathcal{L})$ *and therefore are also detected in* $\mathrm{Pic}(Q(2))$*. The image of* $S^0\langle det\rangle$ *in* $\mathrm{Pic}(Q(2))$ *is given by* $G(0,2,4)$*. The image of* P *is given by* $G(48, \frac{\psi_d(\Delta^2)}{\Delta^2}, 2^{24})$*, where* ψ_d *is the Atkin-Lehner involution.*

An important consequence of this theorem is that, we have now produced non-trivial elements in $\mathrm{Pic}(Q(2))$.

1.3 Outline

In Chapter 2, we recall the necessary background material. In Section 2.1, we discuss the notion of descent. In Section 2.2, we recall the definition of Picard spectra and describe a few results. In Section 2.3, we recall the definition of elliptic curves and modular forms. In Section 2.4, we recall the definition of the spectrum $Q(l)$. In Section 2.5, we recall the definition of Morava stabilizer groups. In Section 2.6, we recall some well known results on the Picard groups of the $K(n)$-local category of spectra.

In Chapter 3, the main result we prove is Theorem 3.3, which is a more detailed version of Theorem 1.1. In Section 3.1, we recall results of [5, 21] which allow us to compute the right adjoint G of F. In Section 3.2, we illustrate our general strategy by recalling the height one case discussed in [20]. In Section 3.3, we apply the strategy to the height two case and prove Theorem 3.3.

In Chapter 4, we prove detection theorems that help us decide when an element is invertible. In Section 4.1, we recall Theorem 4.8 and Theorem 4.9 which were originally proved in [20]. In Section 4.2, we prove Theorem 4.10 and Theorem 4.11 analogous to the height one case.

In Chapter 5, we restrict to the case $l = 2$ and $p = 3$ and we compute the Picard group of the limit $\mathcal{L}$ using a Bousfield-Kan spectral sequence. In Section 5.1, we describe the E_1-page of the spectral sequence. In Section 5.2, we compute the E_2-page of the spectral sequence and we finish the the proof of Theorem 5.1 and deduce Corollary 5.5. Next, we discuss the group structure of $\mathrm{Pic}(\mathcal{L})$ in Section 5.3. Finally, in Section 5.4, we prove Theorem 5.8.

Chapter 2

Background

In this chapter, our main goal is to introduce the notation and essential tools required for this thesis. The primary tool at the heart of this thesis is called *descent*. In Section 2.1, we begin by addressing the concept of descent, exploring its implications, and mentioning a few well-known results that will become valuable later in the thesis. Next, in Section 2.2, we delve into the topic of Picard spectra, following closely the approach outlined in [31].

Moving forward, we focus on the computational prerequisites and tools. Since our primary focus is on height two, in Section 2.3, we dive into the realm of elliptic curves and modular forms. Here, we also introduce the concept of level structures, focusing specifically on $\Gamma_0(2)$ level structures. Further, in Section 2.4, we revisit the construction of the spectrum $Q(l)$. We then narrow our focus to the case where p is 3 and l is 2. In this case, we provide a few formulas and number-theoretic results that will be helpful in our computations in Chapter 5. Since, many of our computations at height 2 are governed by the cohomology of the Morava stablizer group, we briefly recall the relevant theory of Morava stabilizer groups in Section 2.5. Finally, in Section 2.6, we recall the definition of the Picard groups of the $K(n)$-local category of spectra and discuss a few well known results. This section will be primarily used in Chapter 5 to relate the Picard group of $Q(2)$ to the Picard group of the $K(2)$-local category of spectra.

2.1 Descent and module categories

Descent theory has wide-ranging applications in algebraic geometry, homotopy theory, and topology. Descent involves constructing a complex object by assembling simpler ones, requiring that the simpler objects adhere to specific compatibility conditions called the cocycle conditions in order to glue together.

A classical example of this strategy is the case of vector bundles over a manifold M. If the manifold M is not contractible then the vector bundles on M can be hard to classify. However, manifolds admit open covers $\{\mathcal{U}_\alpha\}$, where each $\mathcal{U}_\alpha$ is contractible. On each $\mathcal{U}_\alpha$, any vector bundle V on M restricts to a trivial vector bundle V_α, i.e. $V_\alpha \simeq \mathcal{U}_\alpha \times \mathbb{R}^n$. Since the vector bundles V_α on $\mathcal{U}_\alpha$ arise from a vector bundle on M, there is also gluing data on the intersections, given by the gluing maps $\phi_{\alpha\beta} : V_\alpha|_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta} \to V_\beta|_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta}$. These maps also satisfy a cocycle condition given by $\phi_{\alpha\gamma} = \phi_{\beta\gamma} \circ \phi_{\alpha\beta}$ on the intersection $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$. This allows us to think of a vector bundle V as a collection of trivial vector bundles V_α together with data of gluing maps $\phi_{\alpha\beta}$ which satisfy the cocycle condition. In fact every vector bundle arises this way. This strategy also extends to quasicoherent sheaves on a scheme, using affine open covers instead of general open covers. Further, we can generalize it to the framework of $\mathbb{E}_\infty$-rings. Below, we establish the framework for studying descent for module categories of $\mathbb{E}_\infty$-rings.

Let $\mathcal{C}$ be a presentable symmetric monoidal stable ∞-category and $\mathcal{I}$ be a simplicial set. Let $\mathcal{P}r^L$ be the ∞-category of presentable ∞-categories. By $\mathrm{CAlg}(\mathcal{C})$, we denote the ∞-category of commutative algebras in $\mathcal{C}$. There is a functor

$$\mathrm{Mod} : \mathrm{CAlg}(\mathcal{C}) \to \mathcal{P}r^L \tag{2.1}$$

which sends A to $\mathrm{Mod}_{\mathcal{C}}(A)$, the category of A-modules in $\mathcal{C}$, and a map $f : A \to B$ to the relative tensor product functor $f^* : \mathrm{Mod}_{\mathcal{C}}(A) \to \mathrm{Mod}_{\mathcal{C}}(B)$, which takes an A-module M to the B-module $B \otimes_A M$ [13, 25].

For any map of commutative algebras $f : A \to B$, the relative tensor product functor f^* satisfies two crucial properties that will be relevant in this thesis. Firstly, f^* is a symmetric monoidal functor, and therefore the functor Mod lands in the ∞-subcategory $\mathrm{Cat}^{\otimes}$, the ∞-category of symmetric monoidal ∞-categories. Secondly, f^* has a right adjoint, which is the forgetful functor denoted as f_*. This property will allow us to use the results in [21] in Chapter 3.

Suppose that $R \in \mathrm{CAlg}(\mathcal{C})$ is a limit of an $\mathcal{I}$-diagram of rings

$$\mathcal{R} : \mathcal{I} \to \mathrm{CAlg}(\mathcal{C}). \tag{2.2}$$

Then, we can apply the functor $\mathrm{Mod} : \mathrm{CAlg}(\mathcal{C}) \to \mathcal{P}r^L$ to get a cone from $\mathrm{Mod}_{\mathcal{C}}(R)$ to an $\mathcal{I}$-diagram of module categories

$$\mathcal{D} : \mathcal{I} \to \mathcal{P}r^L. \tag{2.3}$$

Since, any cone factors throught the limit, we get a functor

$$F : \mathrm{Mod}_{\mathcal{C}}(R) \to \lim \mathcal{D}. \tag{2.4}$$

Question 1. *Is the functor F in (3.3) an equivalence?*

This question was investigated in the case $\mathcal{I} = BG$ and $R^{hG} \simeq \lim\limits_{BG} R$ by [14], where BG is the classifying space of a finite group G, and the map $R^{hG} \to R$ is a G-Galois extension in the sense of Rognes [35].

Theorem 2.2 ([14, Theorem 6.10])**.** *Let $R^{hG} \to R$ be a faithful G-Galois extension with G finite. Then the canonical map*

$$\mathrm{Mod}(R^{hG}) \to (\mathrm{Mod}(R))^G$$

is an equivalence of ∞-categories.

This question was also investigated in the case $\mathcal{I} = \Delta$ and

$$R \simeq \lim_{\Delta} \left\{ S \rightrightarrows S \otimes_R S \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} S \otimes_R S \otimes_R S \overset{\rightarrow}{\underset{\rightarrow}{\rightrightarrows}} \cdots \right\}$$

by [29], where Δ is the simplicial indexing category, and S is a descendable R-algebra in the sense of [29, Definition 2.30].

Theorem 2.3 ([30, Proposition 3.22]). *Let $R \to S$ be a descendable morphism in CAlg($\mathcal{C}$). The the natural functor* $\mathrm{Mod}(R)$ *to the limit*

$$\mathrm{Mod}(R) \to \lim_{\Delta} \left\{ \mathrm{Mod}(S) \rightrightarrows \mathrm{Mod}(S \otimes_R S) \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathrm{Mod}(S \otimes_R S \otimes_R S) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots \right\}$$

is an equivalence.

The methods used to prove these results rely on the Barr-Beck-Lurie monadicity theorem [25, Theorem 4.7.3.5]. In contrast, the methods used in this thesis are based on [5, 21]and do not rely on the Barr-Beck-Lurie monadicity theorem.

2.2 Picard groups and Picard spectra

In this section, we provide a brief introduction to Picard groups and Picard spectra. For a more in-depth analysis, please refer to the detailed introduction in [31]. We begin with the definition of Picard group for a symmetric monoidal category.

Definition 2.4. *Given a presentable symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$, the Picard group of C is the group of isomorphism classes of invertible objects, i.e. $x \in \mathcal{C}$ such that there is a $y \in \mathcal{C}$ satisfying $x \otimes y \simeq \mathbb{1}$. We will denote this group by* $\mathrm{Pic}(\mathcal{C})$.

2.2.1 Picard groups in algebra

We will now define the Picard groups for rings and schemes.

Definition 2.5. *Given a commutative ring R, by* $\mathrm{Pic}(R)$ *we denote the Picard group of the symmetric monoidal category of R-modules* $(\mathrm{Mod}(R), \otimes_R, R)$. *We can further generalise and define the Picard group of a scheme S, denoted* $\mathrm{Pic}(S)$, *as the Picard group of the symmetric monoidal category of Quasi-coherent sheaves over S.*

Note that this definition is consistent with the *classical* definition of the Picard group of a scheme. The main advantage of this definition is that it allows us to define the Picard group for more general objects, such as $\mathbb{E}_\infty$-rings.

Example 2.6. If R is a unique factorization domain, then $\mathrm{Pic}(R)$ is trivial. In particular, $\mathrm{Pic}(R)$ is a measure of the failure of unique factorisation in R.

Definition 2.7. *Given a graded ring R_*, by* $\mathrm{Pic}(R_*)$ *be the Picard group of the symmetric monoidal category* $(\mathrm{Mod}(R_*), \otimes_{R_*}, R_*)$.

Example 2.8. For a graded field $K[x^{\pm 1}]$ with $|x| = n$, the Picard group is a cyclic group of order n, and it is generated by the module $\Sigma K[x^{\pm 1}]$.

2.2.2 Picard groups in homotopy theory

For a homotopy theorist, the first symmetric monoidal category that comes to mind is the category of spectra $(Sp, \wedge, S^0)$. The Picard group in this case is $\mathbb{Z}$, and the generator is given by the circle S^1. For a quick proof, see [31, Example 2.16]. While this example may not be exciting, the chromatic Picard groups and Picard groups of $\mathbb{E}_\infty$-rings are significantly more interesting. These Picard groups have been extensively studied in homotopy theory starting with the work of Hopkins, Mahowald and Sadofsky in [20].

Analogous to the classical case of a ring, we define the Picard group of an $\mathbb{E}_\infty$-ring R.

Definition 2.9. *Given an $\mathbb{E}_\infty$-ring R, by $\mathrm{Pic}(R)$ we denote the Picard group of the symmetric monoidal category of R-modules $(\mathrm{Mod}(R), \otimes_R, R)$.*

It is not true that if $M \in \mathrm{Pic}(R)$ then $M_* \in \mathrm{Pic}(R_*)$. However, we do have a map $\Phi : \mathrm{Pic}(R_*) \to \mathrm{Pic}(R)$, which is constructed by realising the invertible module as the homotopy groups of a spectrum, and observing that the spectrum is in fact invertible.

Definition 2.10. *We say $\mathrm{Pic}(R)$ is algebraic if the map $\Phi : \mathrm{Pic}(R_*) \to \mathrm{Pic}(R)$ is an isomorphism.*

Theorem 2.11 ([1, Theorem 9.1])**.** *The Picard group of R is algebraic when R is one of the following:*

1. *HA, where A is a commutative ring.*
2. *MU/I, where MU is the complex cobordism theory and I is a finitely generated ideal of MU_* for which MU/I is a commutative S-algebra.*
3. *KU and ku, where KU represents the complex toplogical K-theory and ku is the connective cover of KU.*
4. *$KO[1/2]$ and ko, where KO represents the real toplogical K-theory and ko is the connective cover of KO.*
5. *$\widehat{E(1)}$ and $\widehat{E(2)}$, where $\widehat{E(1)}$ and $\widehat{E(2)}$ are the completed Johnson-Wilson spectra at heights 1 and 2 respectively.*
6. *tmf at a prime p, where tmf is the connective spectrum of topological modular forms.*
7. *E_n for any height n and prime p, where E_n denotes the Lubin-Tate spectrum at height n.*

To prove the theorem, the main idea is to first establish algebraicity at the local level i.e proving algebraicity in the case of $R_\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal of R. Subsequently, they use local-global arguments to prove the algebraicity of R. However, these arguments require strong assumptions about the homotopy groups of R are necessary, such as requiring that R_* is connective or that R_0 is a regular local ring.

2.2.3 Picard spectra

Picard spectra are a powerful tool to compute Picard groups of $\mathbb{E}_\infty$-rings. For example, in [31], the authors use Picard spectra to compute the Picard groups of the periodic spectrum of topological modular forms TMF and the nonperiodic and non-connective Tmf. Picard spectra are also better behaved homotopically compared to the Picard group, allowing us to use computational tools such as the homotopy limit spectral sequence to compute the Picard group of a limit.

We begin by recalling the definitions of the Picard space and Picard spectrum for a presentable symmetric monoidal ∞-category, as given in [31].

Definition 2.12 ([31, Definition 2.2.1])**.** *Given a presentable symmetric monoidal ∞-category $(\mathcal{C}, \otimes, \mathbb{1})$, let $\mathcal{P}ic(\mathcal{C})$ denote the ∞-groupoid of invertible objects in C and equivalences between them. $\mathcal{P}ic(\mathcal{C})$ is a group-like $\mathbb{E}_\infty$-space, and thus the delooping of a connective Picard spectrum $\mathfrak{pic}(\mathcal{C})$[32].*

Picard spectrum contains a lot of information about $\mathcal{C}$. Notably, its 0th homotopy group corresponds to the Picard group, $\mathrm{Pic}(\mathcal{C})$. The first homotopy group, $\pi_1(\mathfrak{pic}(\mathcal{C}))$, represents the group of automorphisms of the unit $\mathbb{1} \in \mathcal{C}$, $\pi_0\mathrm{End}(\mathbb{1})^\times$. As for the higher homotopy groups, $\pi_i(\mathfrak{pic}(\mathcal{C}))$, they coincide with the shifted homotopy groups of the endomorphism space of the unit, $\pi_{i-1}\mathrm{End}(\mathbb{1})$ for $i \geq 2$.

Definition 2.13. *Given a $\mathbb{E}_\infty$-ring R, let $\mathcal{P}ic(R)$ denote the Picard ∞-groupoid of the presentable symmetric monoidal ∞-category $(\mathrm{Mod}(R), \otimes_R, R)$. $\mathcal{P}ic(R)$ is a group like $\mathbb{E}_\infty$-space, and thus the delooping of a connective Picard spectrum $\mathfrak{pic}(R)$.*

Picard spectrum of a ring R contains interesting homotopical information about R. In particular we have,

$$\pi_0(\mathfrak{pic}(R)) = \pi_0(R)^\times \tag{2.5}$$

$$\pi_i(\mathfrak{pic}(R)) = \pi_{i-1}(R) \text{ for } i \geq 1 \tag{2.6}$$

Pic does not commute with homotopy limits. We now present a fundamental property of the functor $\mathfrak{pic}$ that underpins the computations in this thesis:

Theorem 2.14 ([31, Proposition 2.2.3])**.** *The functor*

$$\mathfrak{pic} : Cat^{\otimes} \to Sp_{\geq 0}$$

from the ∞-category $Cat^{\otimes}$ of symmetric monoidal ∞-categories to the ∞-category $Sp_{\geq 0}$ of connective spectra, commutes with limits.

This theorem allows us to commute the functor $\mathfrak{pic}$ past the homotopy limit. This, in turn, will allow us to use the homotopy limit spectral sequence to compute the Picard group of a limit.

2.2.4 Infinite loop space structure of $\mathcal{P}ic(R)$

We now shift our attention towards the infinite loop space structure of $\mathcal{P}ic(R)$. Even though, the homotopy groups of $\mathcal{P}ic(R)$ closely resemble the homotopy groups of R, their loop space structure is quite different. In order, to understand the loop space structure we consider a related infinite loop space $GL_1(R)$ [33].

In [33], $GL_1(R)$ is defined as the pullback

$$\begin{tikzcd}
GL_1(R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega^\infty(R) \arrow[d] \\
\pi_0(R)^\times \arrow[r] & \pi_0(R)
\end{tikzcd}$$

One can check that $GL_1(R)$ is a group-like infinite loop space, and thus the delooping of a spectrum of units $gl_1(R)$. The loop space structure of $GL_1(R)$ gives rise to a group structure on the maps into $GL_1(R)$.

Given two maps $f, g : X \to GL_1(R)$, we can define $f \cdot g$ to be the composite

$$X \xrightarrow{\Delta} X \times X \xrightarrow{(f,g)} GL_1(R) \times GL_1(R) \xrightarrow{m} GL_1(R) \tag{2.7}$$

Lemma 2.15. *With notation as above, given an element $a \in \pi_0(X)$. The map $f \cdot g$ on π_0, is given by $a \to f(a)g(a)$.*

Proof. This follows from the definition of $f \cdot g$ in equation (2.7). $\square$

We have an equivalence of infinite loopspaces $GL_1(R) \to \Omega(\mathcal{P}ic(R))$ (see [31, Example 2.2.2]). This allows us to translate between the group structure on maps into $Gl_1(R)$ and maps into $\mathcal{P}ic(R)$. As a consequence we have the following corollary:

Corollary 2.16. *Given two maps $f, g : X \to \mathcal{P}ic(R)$ and an element $a \in \pi_1(X)$. The map $f \cdot g$ on π_1, is given by $a \to f(a)g(a)$.*

2.3 Elliptic curves and Modular forms

In this section we will briefly recall the theory of elliptic curves and modular forms. For an in-depth explanation of these concepts see [24, 36].

2.3.1 Elliptic curves

The central objects of study in this thesis are the elliptic curves and the morphisms between them. Let us start with the definition of an elliptic curve.

Definition 2.17 ([24, Chapter 2])**.** *Let S be a scheme. An elliptic curve (E, p, O) over S is a smooth proper morphism $p : E \to S$ whose fibres are geometrically connected curves of genus 1, together with a section $O \in E(S)$. Often, we will omit the map p and the identity section O from our notation and refer to the elliptic curve as E.*

Now, let us introduce the concept of isogenies, which are the morphisms between elliptic curves.

Definition 2.18 ([24, Chapter 2])**.** *Let (E, p, O) be an elliptic curve over S, and (E', p', O') an elliptic curve over S'. A morphism of elliptic curves $(f, g) : (E, p, O) \to (E', p', O')$ consists of morphisms of schemes $f : S \to S'$ and $g : E \to E'$ which satisfy $g \circ O = O' \circ f$ and make the diagram*

$$\begin{array}{ccc} E & \xrightarrow{p} & S \\ \downarrow g & & \downarrow f \\ E' & \xrightarrow{p'} & S' \end{array}$$

cartesian. When the base scheme is the same, we will assume that g is the identity and we will often omit it from our notation.

2.3.2 Group structure on elliptic curves

Elliptic curves admit a group structure. We will introduce this group structure via the concept of the relative Picard scheme, denoted as $\mathrm{Pic}_{E/S}$. Later, we will also use this notion to define dual isogenies.

Definition 2.19. *The relative Picard functor $Pic_{E/S}$ is defined by*

$$Pic_{E/S}(R) := Pic(E_R) \,/\, Pic(R),$$

where $E_R := E \times_S \mathrm{Spec}R$*. By* $Pic^0_{E/S}(R)$ *we denote the identity component of* $Pic_{E/S}(R)$*.*

It turns out that the identity component of this Picard scheme, $\mathrm{Pic}^0_{E/S}$, happens to be the very elliptic curve we began with!

Theorem 2.20 ([24, Chapter 2])**.** *Given a ring* R *over a scheme* S*, there is a bijection of sets* $E(R) \to Pic^0_{E/S}(R)$*, where* $E(R)$ *are the* R*-points of the elliptic curve* E*.*

Given that the Picard scheme is a group scheme, the isomorphism of sets as stated in Theorem 2.20 permits us to transfer the group structure from the Picard scheme to the elliptic curve. With this group structure in place, we can proceed to construct the multiplication-by-m isogeny, which will be crucial in defining Adams operations on TMF.

Example 2.21. For any integer $m > 0$, we define the multiplication-by-m isogeny

$$[m] : E \to E \tag{2.8}$$

$$P \to \underbrace{P + P + \ldots + P}_{\text{m terms}} \tag{2.9}$$

For $m < 0$, we set $[m](P) = [-m](-P)$. The m-torsion subgroup of E, denoted as $E[m]$, is defined to be the kernel of the map $[m]$.

Next, we introduce the concept of a dual isogeny, which will be essential for defining the Atkin-Lehner operator on $TMF_0(l)$.

Definition 2.22. *Let* $f : E \to E'$ *be an isogeny. Then* $\hat{f} = \mathrm{Pic}(f) : \mathrm{Pic}^0_{E'/S} \simeq E' \to \mathrm{Pic}^0_{E/S} \simeq E$ *is called the dual isogeny.*

We will now delve deeper into isogenies and understand some of their fundamental properties. We will mainly state the theorems and we refer the reader to [24] for the proofs of these statements. The first theorem we state is that every isogeny is either trivial (represented as the zero map) or a finite locally free homomorphism.

Theorem 2.23 ([24, Chapter 2])**.** *Let* $f : E \to E'$ *be an isogeny. Then* f *is either* 0 *or a finite locally free homomorphism.*

In particular, every isogeny except the constant is a finite map of elliptic curves. This allows us to define the notion of a degree for the non-zero isogenies. When we compose the isogeny with its dual (denoted as $\hat{f}$), we obtain the multipication-by-[$\deg(f)$] map defined in (2.21).

Theorem 2.24 ([24, Chapter 2])**.** *Let* $f : E \to E$ *be a non-constant isogeny. Then* $\hat{f} \circ f = [\deg(f)]$*, where* $\deg(f)$ *is the degree of the finite map* f*.*

We end this section by stating a theorem about the m-torsion group of an elliptic curve, $E[m]$. Since $\deg(f) = \deg(\hat{f}) = m$, Theorem 2.24 gives us that the degree of $[m]$ is m^2. Further, Theorem 2.25 below tells us that when m is invertible, etale locally $E[m]$ behaves exactly like $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$.

Theorem 2.25 ([24, Chapter 2])**.** *Let* E *be an elliptic curve. The map* $[m] : E \to E$ *is finite locally free of rank* m^2*. If* m *is invertible, its kernel* $E[m]$ *is finite etale over* S*, locally for the etale topology on* S *isomorphic to* $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$*.*

2.3.3 Weierstrass curves

In this subsection, we will explore Weierstrass curves, which offer a computationally convenient representation of elliptic curves. The following theorem, Theorem 2.26, provides a concrete description of elliptic curves Zariski locally.

Theorem 2.26 ([24, Chapter 2, Section 2.2])**.** *For every elliptic curve E, we can find a Zariski covering $S = \bigcup_i U_i$ with $U_i = Spec\ R_i$ so that $E \times_S U_i$ is a Weierstrass curve, i.e., it can be embedded into $\mathbb{P}^2_{R_i}$ and is cut out by an equation*

$$y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3. \tag{2.10}$$

Here, $a_1, a_2, a_3, a_4, a_5, a_6 \in R_i$, and the coordinates (x, y, z) are homogeneous coordinates in the projective space $\mathbb{P}^2_{R_i}$.

For the remainder of the section, we will assume we are working over a affine neighbourhood $\mathrm{Spec}(R)$ and that the Weierstrass curve is given by the equation (2.10). In affine coordinates, we get

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \tag{2.11}$$

However, a single elliptic curve E can admit various Weierstrass equations. By [24, Chapter 2, Section 2.2], any two Weierstrass equations for E are related by a linear change of variables of the form

$$\begin{aligned} x &= u^2x' + r, \\ y &= u^3y' + su^2x' + t, \end{aligned} \tag{2.12}$$

where $u, r, s, t \in R$ and u is a unit.

Given that our computations in this thesis will be at the prime 3, we can assume that 2 is invertible. When 2 is invertible, we can apply the substitution

$$y \to \frac{(y - a_1x - a_3)}{2}$$

to transform equation (2.11) into the form

$$y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6, \tag{2.13}$$

where $b_2 = a_1^2 + 4a_2$, $b_4 = 2a_4 + a_1a_3$, and $b_6 = a_3^2 + 4a_6$.

Definition 2.27. *For every Weierstrass curve, we can define the following quantities:*

$$\begin{aligned} b_8 &:= b_2b_6 - b_4^2 \\ c_4 &:= b_2^2 - 24b_4 \\ c_6 &:= -b_2^3 + 36b_2b_4 - 216b_6 \\ \Delta &:= -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6 \\ \omega &:= \frac{dx}{2y + a_1x + a_3} \\ j &:= \frac{c_4^3}{\Delta} \end{aligned} \tag{2.14}$$

We can verify that $\Delta = \frac{c_4^3 - c_6^2}{1728}$. This quantity is referred to as the **discriminant** *of the Weierstrass curve. Furthermore, ω denotes the* **invariant differential** *associated with the Weierstrass equation. Lastly, j is called the* **j-invariant** *of the elliptic curve.*

These parameters, specifically c_4, c_6, ω, and Δ, depend on the choice of the Weierstrass equation used to represent an elliptic curve. However, since our focus lies on the curve E rather than its specific Weierstrass representation, it is essential to understand how these quantities transform under a change of coordinates as shown in equation (2.17).

We summarize these transformations below in table 2.1.

$u^4 c_4'$	$=$	c_4
$u^6 c_6'$	$=$	c_6
$u^{12}\Delta'$	$=$	Δ
j'	$=$	j
$u^{-1}\omega'$	$=$	ω

Table 2.1: Coordinate Transformation Formulae

Looking at the table, we can observe that c_4, c_6, j, and Δ remain nearly unchanged, except for multiplication by a unit.

2.3.4 Modular forms

Modular forms are sections of a specific line bundle over the moduli stack of elliptic curves. So, we begin by introducing the moduli stack of elliptic curves.

Definition 2.28. *The moduli stack of elliptic curves $\mathcal{M}_{ell}$ associates to every ring R, the groupoid of elliptic curves E over R.*

The moduli stack $\mathcal{M}_{ell}$ is not affine due to the fact that elliptic curves possess non-trivial automorphisms. Next, we delve into elliptic curves with level structures and their associated moduli stacks.

Definition 2.29. *An elliptic curve with a $\Gamma_0(l)$ level structure is a pair (E, H) where H is a cyclic order l subgroup.*

Remark 2.30. An elliptic curve with a $\Gamma_0(l)$ level structure (E, H) is in one-to-one correspondence with degree l isogenies with kernel H.

Definition 2.31. *The moduli stack of elliptic curves with $\Gamma_0(l)$ level structure $\mathcal{M}_0(l)$ associates to every ring R, the groupoid of pairs (E, H), where E is an elliptic curve over R and $H \subset E(R)$ is a cyclic order l subgroup.*

In Definition 2.27, we introduced the one-form ω. The form ω behaves well with respect to base change and defines a line bundle over $\mathcal{M}_{ell}$.

Definition 2.32. *A modular form of weight n is a section of the line bundle $\omega^{\otimes n}$.*

Example 2.33. The terms c_4, c_6, j and Δ defined in definition 2.27 are modular forms of weight $4, 6, 12$ and 0 respectively. Base changed to $\mathbb{C}$, c_4 and c_6 are constant multiples of the modular forms G_4 and G_6

respectively. Here G_4 and G_6 are the Eisenstein series of weight 4 and 6 respectively given by the formulae:

$$G_4(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m+n\tau)^4} \tag{2.15}$$

$$G_6(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m+n\tau)^6}. \tag{2.16}$$

Definition 2.34. *A modular form with $\Gamma_0(l)$ level structure and weight n is a section of the line bundle $\omega^{\otimes n}$ on the stack $\mathcal{M}_0(l)$.*

Given an elliptic curve with a $\Gamma_0(l)$ level structure, we can simply forget the level structure and get back an elliptic curve. In particular, we have a forgetful map $\mathcal{M}_0(l) \to \mathcal{M}_{ell}$, which induces an inclusion of the ring of modular forms into the ring of modular forms with level structure. However, the converse is not true. The ring of modular forms with level structure can contain elements that are not modular forms (see Section 2.3.5).

2.3.5 Modular forms with $\Gamma_0(2)$ level structures

In this section, we will briefly discuss the ring of modular forms equipped with $\Gamma_0(2)$ level structures, denoted $MF_0(2)$. For a more in-depth discussion we refer the reader to [8]. For the purposes of this section we work over the ring $\mathbb{Z}[\frac{1}{2}]$.

As seen in the previous section, an elliptic curve can be represented as

$$y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6$$

as given in equation (2.13).

An elliptic curve with a $\Gamma_0(2)$ level structure, is equivalent to the information of an order 2 point on the ellipitc curve. Consider a point of order 2, denoted as (x, y), on the elliptic curve. By applying the group law for elliptic curves, we can show that y must equal zero. Consequently, the presence of $\Gamma_0(2)$ provides us with a root, denoted as e, of the cubic polynomial $4x^3 + b_2x^2 + 2b_4x + b_6$. Therefore, we can factorize the cubic polynomial and express the equation as

$$y^2 = 4(x-e)(x^2 + \alpha x + \beta),$$

where

$$\begin{aligned} b_2 &= 4(\alpha - e), \\ b_4 &= 2(\beta - e\alpha), \\ b_6 &= -4e\beta. \end{aligned}$$

Implementing the coordinate change $x \to x - e$, we arrive at the equation

$$y^2 = 4x(x^2 + q_2x + q_4),$$

where

$$q_2 = 2e + \alpha,$$
$$q_4 = e^2 + e\alpha + \beta.$$

By [36, Proposition 3.1.], any two Weierstrass equations for E are related by a linear change of variables of the form

$$\begin{aligned} x &= u^2 x', \\ y &= u^3 y', \end{aligned} \tag{2.17}$$

where u is a unit in R.

The moduli stack $\mathcal{M}_0(2)$ is not an affine scheme due to the presence of non-trivial automorphisms. If we instead consider the moduli stack of elliptic curves that preserve the invariant differential ω, denoted $\mathcal{M}_0^1(2)$. In this case, there are no non-trivial automorphisms that preserve both the $\Gamma_0(2)$ structure and the differential ω. As a result, we can conclude that the stack $\mathcal{M}_0^1(2)$ is an affine scheme.

In particular, $\mathcal{M}_0^1(2)$ can be described as $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}, q_2, q_4, \Delta^{-1}])$. Therefore, $MF_0(2)[\frac{1}{2}] = \mathbb{Z}[\frac{1}{2}, q_2, q_4, \Delta^{-1}]$. In particular, q_2 and q_4 are modular forms with a $\Gamma_0(2)$ level structure, and together, they generate all modular forms with $\Gamma_0(2)$ level structure.

Given an ellipitc curve with level structure, we can simply forget the level structure and get back an elliptic curve. This forgetful map induces an inclusion of the ring of modular forms into the ring of modular forms with level structure. Since, q_2 and q_4 generate all the modular forms with level structure, we can rewrite any modular form in terms of q_2 and q_4. These formulae can be readily deduced from [36, Chapter III.1 Weierstrass Equations, 8] and we record them below.

$$c_4 = 16q_2^2 - 48q_4 \tag{2.18}$$

$$\Delta = 16q_4^2(q_2^2 - 4q_4) \tag{2.19}$$

$$j = \frac{c_4^3}{16q_4^2(q_2^2 - 4q_4)} \tag{2.20}$$

2.4 Spectrum $Q(l)$

In this section, we will provide a brief overview of the construction of the spectrum denoted as $Q(l)$, which was originally introduced by Behrens in [8]. The primary purpose behind constructing this spectrum was to provide a modular explanation for the duality resolution of the $K(2)$-local sphere in [15]. For the purposes of this section fix a prime p, and assume all the spectra are $K(2)$-localised. Furthermore, let l denote a topological generator of the p-adic units $\mathbb{Z}_p^\times$.

The Goerss-Hopkins-Miller theorem produces a sheaf of E_∞-ring spectra $\mathcal{O}_{GHM}^{top}$ on the small étale site of $\mathcal{M}_{ell}$, where $\mathcal{M}_{ell}$ is the moduli stack of elliptic curves. Further, by [10], the sheaf $\mathcal{O}_{GHM}^{top}$ can be extended to a sheaf $\mathcal{O}^{top}$ which is functorial with respect to isogenies of elliptic curves of invertible degree.

The spectrum $Q(l)$ is the global sections $\Gamma\left(\mathcal{M}_\bullet, \mathcal{O}^{top}\right)$ of a derived semi simplicial stack $\mathcal{M}_\bullet$, defined as

$$\mathcal{M}_\bullet \coloneqq \lim(\mathcal{M}_{ell} \overleftarrow{\overleftarrow{\quad}} \mathcal{M}_0(l) \amalg \mathcal{M}_{ell} \overleftarrow{\overleftarrow{\overleftarrow{\quad}}} \mathcal{M}_0(l)), \tag{2.21}$$

where $\mathcal{M}_0(l)$ represents the moduli stack of elliptic curves with $\Gamma_0(l)$ level structures.

All the maps in the semi simplicial stack arise from three maps denoted, ψ_d, ϕ_f and $\psi_{[l]}$ respectively. The map $\psi_{[l]}$ is given by

$$\begin{aligned}\psi_{[l]} :& \mathcal{M}_{ell} \to \mathcal{M}_{ell} \\ &(C) \to (C/C[l]),\end{aligned} \tag{2.22}$$

where C is an elliptic curve and $C[l]$ is the group of points of order l. The map ψ_d is given by

$$\begin{aligned}\psi_d :& \mathcal{M}_0(l) \to \mathcal{M}_0(l) \\ &(C, H) \to (C/H, Ker(\hat{\phi}_H)),\end{aligned} \tag{2.23}$$

where (C, H) is an elliptic curve with an order l subgroup H and $\hat{\phi}_H$ is the dual isogeny corresponding to the map $\phi_H : C \to C/H$. Finally, the map ϕ_f is given by

$$\begin{aligned}\phi_f :& \mathcal{M}_0(l) \to \mathcal{M}_{ell} \\ &(C, H) \to C.\end{aligned} \tag{2.24}$$

We are now ready to introduce the face maps within the semi-simplicial diagram. The face maps d_i : $\mathcal{M}_0(l) \amalg \mathcal{M}_{ell} \to \mathcal{M}_{ell}$ are given by

$$d_0 = \phi_f \circ \psi_d \amalg \psi_{[l]}, \tag{2.25}$$

$$d_1 = \phi_f \amalg \mathbb{1}_{\mathcal{M}_{ell}}. \tag{2.26}$$

To help with notation we will sometime use

$$\phi_q := \phi_f \circ \psi_d. \tag{2.27}$$

The face maps $d_i : \mathcal{M}_0(l) \to \mathcal{M}_0(l) \amalg \mathcal{M}_{ell}$ are given by

$$d_0 = \psi_d, \tag{2.28}$$

$$d_1 = \phi_f, \tag{2.29}$$

$$d_2 = \mathbb{1}_{\mathcal{M}_0(l)}. \tag{2.30}$$

The global sections of $\mathcal{M}_{ell}$ and $\mathcal{M}_0(l)$ are called TMF and $TMF_0(l)$ respectively. Since, all the maps in (2.21) arise from isogenies of elliptic curves, we can take global sections of $\mathcal{O}^{top}$ to get the semi cosimplicial diagram below in (2.31).

$$Q(l) := \lim(TMF \rightrightarrows TMF_0(l) \times TMF \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} TMF_0(l)\ ) \tag{2.31}$$

The face maps $d_i : TMF \to TMF_0(l) \times TMF$ are given by

$$d_0 = (\psi_d \circ \phi_f) \oplus \psi_{[l]}, \tag{2.32}$$

$$d_1 = \phi_f \oplus \mathbb{1}_{TMF}. \tag{2.33}$$

The face maps $d_i : TMF_0(l) \times TMF \to TMF_0(l)$ are given by

$$d_0 = \psi_d \circ p_1, \tag{2.34}$$
$$d_1 = \phi_f \circ p_2, \tag{2.35}$$
$$d_2 = p_1, \tag{2.36}$$

where p_1 and p_2 represent projections on to $TMF_0(l)$ and TMF respectively.

For computations, we will restrict to the case p is 3 and l is 2. In this case, the spectrum $Q(2)$ is closely related to the $K(2)$-local sphere, $L_{K(2)}S^0$ and in fact satisfies a cofiber sequence [8]

$$DQ(2) \to L_{K(2)}S^0 \to Q(2), \tag{2.37}$$

where $DQ(2)$ is the $K(2)$-local Spanier-Whitehead dual of $Q(2)$.

For the computations in this thesis, we will require information on the zeroth homotopy group of TMF and the homotopy groups of $TMF_0(l)$. Since we are working $K(2)$-locally, the zeroth homotopy group of TMF is given by [15]

$$\pi_0(TMF) = \mathbb{Z}_3[[j]], \tag{2.38}$$

where j is the *j-invariant* defined in Definition 2.27.

Similarly, the homotopy groups of $TMF_0(2)$ are given by [8, 15]

$$\pi_*(TMF_0(2)) = \mathbb{Z}_3[[x]][q_2, q_4^{\pm}], \tag{2.39}$$

where $x := \frac{q_2^2}{4q_4}$ with q_2 and q_4 as defined in Section 2.3.5. The degree of q_2 and q_4 are 4 and 8 respectively. In particular, the homotopy groups of $TMF_0(2)$ are 8-periodic. Additionally, the following formulas in equations (2.40) and (2.41), derived in [8], describe the action of the Atkin-Lehner map ψ_d on q_2 and q_4:

$$\psi_d(q_2) = -2q_2 \tag{2.40}$$
$$\psi_d(q_4) = q_2^2 - 4q_4 \tag{2.41}$$

Using (2.18)-(2.41), we establish Lemma 2.35, which provides the required formulas used in the proof of Theorem 5.2.

Lemma 2.35. *With j, q_2, q_4 and ψ_d as in equations 2.20, 2.40 and 2.41, the following formulae hold.*

(1) $j = 16^2\left(\frac{27}{4} - \frac{81}{4}x + \frac{63}{4}x^2 - \frac{1}{4}\left(\sum_{i=3}^{\infty} x^i\right)\right)$*, with x as in equation (2.39).*

(2) $\psi_d(x) = \frac{-x}{1-x}$.

(3) $j^i \equiv (-1)^i \left(\frac{x^3}{1-x}\right)^i \pmod 3$.

(4) $\psi_d(j)^i \equiv \left(\frac{x^3}{(1-x)^2}\right)^i \pmod 3$.

Proof. Equation (1) is derived by using formulae (2.18) and (2.20) for c_4 and j respectively, followed by straightforward algebraic manipulations.
Equation (2) is derived by applying the formulae for the action of the Atkin-Lehner operator in (2.40) and (2.41).

Equation (3) is obtained by reducing equation (1) modulo 3.
Equation (4) is derived by combining equations (2) and (3), followed by simple algebraic manipulations. ☐

Remark 2.36. Equation (2) in 2.35, implies that $\psi_d : \mathbb{Z}_3[[x]] \to \mathbb{Z}_3[[x]]$ is a continous map with respect to the $\mathfrak{m}$-adic topology i.e., $\psi_d(\mathfrak{m}) \subset \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal in $\mathbb{Z}_3[[x]]$.

While proving Theorem 5.1, we will find it necessary to explore the multiplicative properties of the ring $\mathbb{Z}_3[[x]]$, which is the 0th homotopy group of $TMF_0(2)$. To that end, we wrap up this section by introducing a lemma and some definitions that revolve around the group of units of the ring $\mathbb{Z}_3[[x]]$. These will play a crucial role in our proof of Theorem 5.1.

To begin, we introduce a lemma that characterizes which elements belong to the group of squares in the unit group $\mathbb{Z}_3[[x]]^\times$.

Lemma 2.37. *An element $u \in Z_3[[x]]$ has a square root if and only if $u \equiv 1 \pmod{(3, x)}$*

Proof. This follows by Hensel's lemma. We consider $f(x) = x^2 - \sum_{i=0}^{\infty} a_i j^i$ a monic polynomial in $R[x]$. Looking at this polynomial modulo the ideal $(3, j)$, we get $\widetilde{f}(x) = x^2 - a_0$ over $\mathbb{F}_3[x]$. This breaks into linear factors if and only if $a_0 \equiv 1 \bmod 3$.

Assuming $a_0 \equiv 1 \bmod 3$, by Hensel's lemma we can lift these factors to $R[x]$. So the only elements that have square roots are power series whose constant terms is congruent to 1 modulo 3. ☐

We also introduce a few definitions that will help streamline our notation. First, we define the concept of an absolute value.

Definition 2.38. *Let $v \in \mathbb{Z}_3[[x]]^\times$. Then we define the absolute value of v, denoted*

$$|v| = \begin{cases} v, & \text{if } v \text{ is a square} \\ -v & \text{otherwise} . \end{cases} \tag{2.42}$$

Remark 2.39. By Lemma 2.37, $|v|$ is always a square.

Given a square $u = v^2$ in $\mathbb{Z}_3[[x]]^\times$, we have two choices of a square root i.e. v and $-v$. We define the positive square root as follows:

Definition 2.40. *Let $u \in \mathbb{Z}_3[[x]]$ be a square. Then we define $\sqrt{u} := |v|$, where $v^2 = u$.*

2.5 Morava stablizer group

In this section, we will provide a brief introduction to the Morava stabilizer group at height n, denoted as $\mathbb{G}_n$. For a more detailed introduction, see [19]. There are many ways to think of the Morava stabilizer group. Nevertheless, we will adopt the approach in [19], as it enables us to efficiently introduce the necessary objects for our computations. At the end of the section, we will specialize to the case $n = 2$ and $p = 3$.

We begin by introducing a non-commutative algebra O_n, that will allow us to define the Morava stabilizer group $\mathbb{G}_n$.

Definition 2.41. *Let O_n be the non-commutative algebra over $W(\mathbb{F}_{p^n})$, the ring of Witt vectors for the field $\mathbb{F}_{p^n}$, generated by an element S subject to the relations:*

$$S^n = p, \tag{2.43}$$

$$Sw = w^\sigma S, \tag{2.44}$$

for each $w \in W(\mathbb{F}_{p^n})$, where w^σ is the result of applying the lift of Frobenius on w. In particular,

$$O_n = W(\mathbb{F}_{p^n})[S]/(S^n = p, Sw = w^\sigma S). \tag{2.45}$$

We can now define the Morava stabilizer group.

Definition 2.42. *The small Morava stabilizer group $\mathbb{S}_n$, is defined as the group of units in O_n. The Morava stabilizer group $\mathbb{G}_n$ is the semidirect product $\mathbb{G}_n := \mathbb{S}_n \rtimes Gal(\mathbb{F}_{p^n} : \mathbb{F}_p)$.*

The Goerss-Hopkins-Miller theorem, tells us that the group $\mathbb{G}_n$ acts on the Lubin-Tate spectrum E_n via $\mathbb{E}_\infty$-ring maps. The homotopy fixed points of this action is precisely the $L_{K(n)}S^0$. In particular, $\mathbb{G}_n$ is the $K(n)$-local Galois group of the extension $L_{K(n)}S^0 \to E_2$, as described in [35].

We will now introduce the determinant map, denoted as det, which maps $\mathbb{G}_n$ to the group of p-adic units, $\mathbb{Z}_p^\times$. This map was used in [16, 23] to construct a generator for invertible elements in $K(2)$-local spectra. We will need this map to understand the image of $S^0\langle det\rangle$ in $\mathrm{Pic}(Q(2))$.

To begin, we first observe that O_n is a free $W(\mathbb{F}_{p^n})$-module of rank n with a basis given by the elements $1, S, \ldots, S^{n-1}$. Therefore, every element x in O_n can be uniquely expressed as:

$$x = a_0 + a_1 S + \ldots + a_{n-1}S^{n-1}, \tag{2.46}$$

where $a_0, a_1, \ldots, a_{n-1} \in W(\mathbb{F}_{p^n})$.

The canonical right action of $\mathbb{S}_n$ on O_n yields a homomorphism

$$\mathbb{S}_n \to \mathrm{Aut}(O_n) \cong \mathrm{GL}_n(W(\mathbb{F}_{p^n})). \tag{2.47}$$

We will now define the determinant map in equation (2.48).

Definition 2.43. *After composing the action map (defined in (2.47)) with the matrix determinant*

$$\mathbb{S}_n \to GL_n(W(\mathbb{F}_{p^n})) \xrightarrow{determinant} W(\mathbb{F}_{p^n})^\times,$$

the resulting image is precisely $\mathbb{Z}_p^\times \subseteq W(\mathbb{F}_{p^n})^\times$. Consequently, we get the determinant homomorphism

$$det : \mathbb{G}_n \to Gal(\mathbb{F}_{p^n}/\mathbb{F}_p) \times \mathbb{Z}_p^\times \xrightarrow{proj} \mathbb{Z}_p^\times. \tag{2.48}$$

We will now restrict to the case where the height is 2 and prime is 3. In this case, we define two elements in $\mathbb{G}_2$ that will play a crucial role in the computations.

Definition 2.44.

1. *We use ω to denote a fixed lift of a chosen primitive 8th root of unity in $W(\mathbb{F}_9)$. This is an element of order 8 in the Morava stabilizer group $\mathbb{G}_2$.*

2. *We use a to denote an element of order 3 in $\mathbb{G}_2$ given by*

$$a = -\frac{1}{2}(1 + \omega S).$$

Following [15], we can use the two elements ω and a (defined in Definition 2.44) to define two finite subgroups of $\mathbb{G}_2$. These subgroups are closely linked to the spectra TMF and $TMF_0(2)$ as explained below in equation (2.49).

Definition 2.45. *Let $\phi \in Gal(\mathbb{F}_9/\mathbb{F}_3)$ be the Frobenius. We define the following finite groups:*

1. $G_{24} := \langle a, \omega^2, \phi \rangle$
2. $D_8 := \langle \omega^2, \phi \rangle$

Since G_{24} and D_8 are subgroups of $\mathbb{G}_2$, we can take the homotopy fixed points of E_2 with respect to these subgroups. In this case, we have the following (see [8]):

$$\begin{aligned} E_2^{hG_{24}} &\simeq TMF \\ E_2^{hD_8} &\simeq TMF_0(2) \end{aligned} \tag{2.49}$$

By [15], we know the homotopy groups of both $E_2^{hG_{24}}$ and $E_2^{hD_8}$. Further, we also know their completed E_2-homology. In particular, we have

$$\begin{aligned} E_{2_*}(E_2^{hG_{24}}) &\simeq \mathrm{map}^c(\mathbb{G}_2/G_{24}, E_{2_*}) \\ E_{2_*}(E_2^{hD_8}) &\simeq \mathrm{map}^c(\mathbb{G}_2/D_8, E_{2_*}) \end{aligned} \tag{2.50}$$

2.6 Chromatic Picard groups

In this section, we explain some well known results regarding the chromatic Picard groups, which were first introduced in [20]. In the chromatic picture, we have the symmetric monoidal category of $K(n)$-local spectra, denoted as $(Sp_{K(n)}, \hat{\otimes}, L_{K(n)}S^0)$. Given two $K(n)$-local spectra, X and Y, their tensor product in this symmetric monoidal category $X \hat{\otimes} Y := L_{K(n)}(X \otimes Y)$. It is worth noting that the smash product of two $K(n)$-local spectra might not be $K(n)$-local itself, which is why the subsequent $K(n)$-localisation step is crucial. We will denote the Picard group of $(Sp_{K(n)}, \hat{\otimes}, L_{K(n)}S^0)$ by Pic_n.

We will now explore several well-known results regarding these Picard groups. We begin by discussing two central results from [20]. The first result concerns the computation of Pic_1.

Theorem 2.46 ([20, Theorem 3.3, Proposition 2.7])**.** *At odd primes p,*

$$\mathrm{Pic}_1 \simeq \mathbb{Z}_p \oplus \mathbb{Z}/2(p-1).$$

At the prime 2*,*

$$\mathrm{Pic}_1 \simeq \mathbb{Z}_2^\times \oplus \mathbb{Z}/4.$$

In this thesis, we will delve into the details of the odd prime case in Theorem 2.46. We will explain how we can interpret this theorem as a descent result for the Picard spectrum, as shown in Theorem 4.9.

The second result from [20] we discuss, relates the Picard group Pic_n with the group cohomology of the Morava stabilizer group $\mathbb{G}_n$.

Theorem 2.47 ([20, Proposition 7.5])**.** *There is a map*

$$\gamma : \mathrm{Pic}_n \to H^1(\mathbb{G}_n, W_{\mathbb{F}_{p^n}}[[u_1, ..., u_{n-1}]]^\times)$$

and γ is an injection if $n^2 \leq 2p-2$ and $p > 2$,where $\mathbb{G}_n$ is the Morava stabiliser group associated to the formal group of height n and $W_{\mathbb{F}_{p^n}}$ are the Witt vectors of $\mathbb{F}_{p^n}$. The cohomology group $H^1(\mathbb{G}_n, W_{\mathbb{F}_{p^n}}[[u_1, ..., u_{n-1}]]^\times)$ is called the **algebraic Picard group***, and is denoted* $\mathrm{Pic}_{n,alg}$.

In general, we do not expect the map γ to be an injection. That is, we expect to have many elements in Pic_n which are not detected by the cohomology of .the Morava stabiliser group.

Remark 2.48. We denote the kernel of the map γ by κ_n. Very little is known about κ_n. In the case of height 1 at prime 2, $\kappa_1 \simeq \mathbb{Z}/2\mathbb{Z}$ with the generator being the $L_{K(1)}DQ$ where DQ is the dual of the *question mark complex*. If we extend the Hopf map $\eta : S^1 \to S^0$ to a map from the Moore spectrum $\widetilde{\eta} : \Sigma M_2 \to S^0$. Then DQ is the cofiber of $\widetilde{\eta}$.

We will now move on to the Picard groups at height 2. At height 2, for primes $p \geq 5$, Hopkins computed the Picard group of the $K(2)$-local category of spectra Pic_2.

Theorem 2.49 (Hopkins)**.** *For primes $p \geq 5$,* Pic_2 *is algebraic and given by* $\mathrm{Pic}_2 \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}/2(p^2 - 1)$.

At the primes 2 and the 3, the computation is significantly more involved. At prime 2, [6] show that the exotic picard group $\kappa_2 \simeq (\mathbb{Z}/8)^2 \times (\mathbb{Z}/2)^3$. The group Pic_2 is currently work in progress.

For the rest of this section, our focus will be on the prime 3 case. In this case, the computation of Pic_2 was completed in two stages. In the first stage, Karmanov computed the algebraic Picard group.

Theorem 2.50 ([23])**.** *At the prime* 3*,*

$$\mathrm{Pic}_{2,alg} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}/16.$$

Finally, in [16], the authors computed the group κ_2 (see Remark 2.48) of *exotic* elements.

Theorem 2.51 ([16, Theorem 1.1])**.** *At the prime* 3*,*

$$\kappa_2 \simeq \mathbb{Z}/3 \times \mathbb{Z}/3.$$

Thus, we get that $\mathrm{Pic}_2 \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}/16 \times \mathbb{Z}/3 \times \mathbb{Z}/3$.

In this thesis, in Theorem 5.8, we will use the results in Theorems 2.50 and 2.51 to compute the image of Pic_2 in the Picard group of the spectrum $Q(2)$. Therefore, we will need a few properties of the generators of Pic_2.

In [23], the author shows that $\mathrm{Pic}_{2,alg}$ is topologically generated by S^1 and $S^0\langle det\rangle$, where $S^0\langle det\rangle$ is the spectrum constructed in [3, 16]. In [16], the authors construct spectra P and Q which generate each of the $\mathbb{Z}/3$ in κ_2.

In Theorem 5.8, we will discuss the images of $S^0\langle det\rangle$ and P. So, we conclude this section by discussing some of the properties of the spectra $S^0\langle det\rangle$ and P.

A crucial property of $S^0\langle det\rangle$ is that the Morava module $E_{2_*}(S^0\langle det\rangle)$ is simply given by twisting the action of G_2 on E_{2_*} by the determinant map defined in (2.48) (for details, refer to [3]). In particular, using the determinant map, we can define the Morava module $E_{2_*} \otimes \mathbb{Z}_3^\times$ with the diagonal G_2-action, and we obtain a G_2-equivariant isomorphism $E_{2_*}(S^0\langle det\rangle) \simeq E_{2_*} \otimes \mathbb{Z}_3^\times$.

Regarding the spectrum P, a key property we will use is the existence of an isomorphism $P \otimes TMF \simeq \Sigma^{48}TMF$, which induces an isomorphism of the Morava modules $\mathrm{map}^c(G_2/G_{24}, (E_2)_*) \to \mathrm{map}^c(G_2/G_{24}, \Sigma^{48}(E_2)_*)$ given by multiplication by Δ^2 (see [16, Theorem 5.5]).

Chapter 3

Construction of Picard elements

Let $\mathcal{C}$ be a presentable symmetric monoidal stable ∞-category and let $\mathcal{I}$ be a simplicial set. Suppose that $R \in \mathrm{CAlg}(\mathcal{C})$ is a limit of an $\mathcal{I}$-diagram of rings

$$\mathcal{R} : \mathcal{I} \to \mathrm{CAlg}(\mathcal{C}). \tag{3.1}$$

Then, we can apply the functor $\mathrm{Mod} : \mathrm{CAlg}(\mathcal{C}) \to \mathcal{P}r^L$ to get a cone from $\mathrm{Mod}_{\mathcal{C}}(R)$ to an $\mathcal{I}$-diagram of module categories

$$\mathcal{D} : \mathcal{I} \to \mathcal{P}r^L, \tag{3.2}$$

where $\mathcal{P}r^L$ is the ∞-category of presentable ∞-categories. Since, any cone factors throught the limit, we get a functor

$$F : \mathrm{Mod}_{\mathcal{C}}(R) \to \lim \mathcal{D}. \tag{3.3}$$

In this thesis, we focus on the case where $\mathcal{I}$ is finite, meaning it consists of a finite number of non-degenerate simplices. In this case, we have the following theorem:

Theorem 3.1 ([26, Theorem 7.2])**.** *With notation as above, for finite diagrams $\mathcal{I}$ the natural functor*

$$F : \mathrm{Mod}_{\mathcal{C}}(R) \to \lim \mathcal{D}$$

is fully faithful.

The adjoint functor theorem (see [28, Prop 5.5.3.13]) ensures the existence of a right adjoint G for the functor F. Since F is fully faithful the right adjoint G is essentially surjective.

This has interesting implications for the Picard group Pic(R). In particular, we can express every element in $\mathrm{Pic}(R)$ as $G(X)$, where X belongs to $\mathrm{Pic}(\lim \mathcal{D})$. However, it is important to note that since G may not necessarily be symmetric monoidal, applying G on $\mathrm{Pic}(\lim \mathcal{D})$ may yield elements in $\mathrm{Mod}(R)$ that are not invertible. So, we think of elements of the form $G(X)$ with $X \in \mathrm{Pic}(\lim \mathcal{D})$ as *potential* candidates for elements in $\mathrm{Pic}(R)$. In Chapter 4, we will give a criterion for when these potential elements are indeed invertible.

For this approach to be effective, it is essential to have a explicit description of the right adjoint G. In the next section, we describe the results of [21] that allow us to compute G. Subsequently, in Section 3.2, we will apply these results in to generate potential elements in $\mathrm{Pic}(Sp_{K(1)})$. These turn out to be precisely the spectra X_λ constructed in [20]. Finally, in Section 3.3, we end the chapter by constructing potential elements in $\mathrm{Pic}(Q(l))$.

3.1 The right adjoint G

Let G denote the right adjoint of the functor F in (3.3). The main goal of this section is to give an explicit formula for the right adjoint G on the objects of $\lim \mathcal{D}$.

Following [21], we begin with the observation that the functor F in (3.3) can be written as a composite

$$\mathrm{Mod}_{\mathcal{C}}(R) \xrightarrow{\Delta} \lim \mathrm{Mod}_{\mathcal{C}}(R)^{ct} \xrightarrow{F^{\mathcal{I}}} \lim \mathcal{D}, \tag{3.4}$$

where $\mathrm{Mod}_{\mathcal{C}}(R)^{ct}$ denotes the constant $\mathcal{I}$-diagram. In [21, Theorem 5.5], the authors construct a functor $G^{\mathcal{I}}$ such that the right adjoint G can be expressed as a composite

$$\lim \mathcal{D} \xrightarrow{G^{\mathcal{I}}} \mathrm{Mod}_{\mathcal{C}}(R)^{\mathcal{I}} \xrightarrow{\lim} \mathrm{Mod}_{\mathcal{C}}(R). \tag{3.5}$$

Here, $\mathrm{Mod}_{\mathcal{C}}(R)^{\mathcal{I}}$ denotes the category of $\mathcal{I}$-diagrams in $\mathrm{Mod}_{\mathcal{C}}(R)$ and lim denotes the limit functor. Therefore, given an object $X \in \lim \mathcal{D}$, we get that $G(X) \simeq \lim G^{\mathcal{I}}(X)$.

Our goal now is to give an explicit description of the functor $G^{\mathcal{I}}$ on the objects of $\lim \mathcal{D}$.
For the purposes of this thesis, it suffices to consider finite $\mathcal{I}$ of dimension at most two. Therefore, for the rest of the chapter, we assume that $\mathcal{I}$ is a finite simplicial set of dimension at most 2.

3.1.1 Description of the objects in $\lim \mathcal{D}$

Before we dive into the description of $G^{\mathcal{I}}$ on objects, we need to understand the objects of $\lim \mathcal{D}$. We begin by recalling the closely related notion of the lax limit of ∞-categories.

Let $q : \mathcal{Y} \to \mathcal{I}$ be a coCartesian fibration classified by the diagram $\mathcal{D}$ (see [28, Theorem 3.2.0.1, Chapter 3]). This coCartesian fibration $q : \mathcal{Y} \to \mathcal{I}$ has the crucial property that the fiber over each vertex $a \in \mathcal{I}$ is precisely the ∞-category $\mathcal{D}(a) \simeq \mathrm{Mod}(\mathcal{R}(a))$.

The lax limit of $\mathcal{D}$, denoted as $\mathrm{Lax}(\mathcal{D})$, is defined to be the ∞-category of sections of the coCartesian fibration q. An object $m \in \mathrm{Lax}(\mathcal{D})$ is given by a section $m : \mathcal{I} \to \mathcal{Y}$. Therefore, it is a family $\{m_\sigma\}$ for each non-degenerate simplex $\sigma \in \mathcal{I}$, satisfying the following conditions:

1. For each 0-simplex $a \in \mathcal{I}$, m_a is an $\mathcal{R}(a)$-module. Here $\mathcal{R}$ is the underlying diagram of rings in (3.1).

2. For each 1-simplex $f : a \to b$ in $\mathcal{I}$, $m_f : \mathcal{R}(f)^* m_a \to m_b$ is a morphism in $\mathrm{Mod}(\mathcal{R}(b))$, where $\mathcal{R}(f)^*$ denotes the functor obtained by tensoring-up along $\mathcal{R}(f)$.

3. For each 2-simplex $\tau \in \mathcal{I}$

$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "h"'] & \arrow[u, phantom, "\Downarrow \tau"] & c \end{tikzcd}$$

m_τ fills the following diagram in $\mathrm{Mod}(\mathcal{R}(c))$

$$\begin{tikzcd}
& \mathcal{R}(g)^*m_b \arrow[dr, "m_g"] \arrow[d, phantom, "\Downarrow m_\tau"] & \\
\mathcal{R}(g)^*\mathcal{R}(f)^*m_a \simeq \mathcal{R}(h)^*m_a \arrow[ur, "\mathcal{R}(g)^*m_f"] \arrow[rr, "m_h"'] & & m_c
\end{tikzcd}$$

We will now describe the objects of $\lim \mathcal{D}$. By [28, Proposition 3.3.3.2], $\lim \mathcal{D}$ can be identified with the full subcategory of $\mathrm{Lax}(\mathcal{D})$ spanned by the coCartesian sections. By the universal property of coCartesian morphisms, m_f is a coCartesian morphism if and only if it is an isomorphism. Therefore, a section $m \in \mathrm{Lax}(\mathcal{D})$ is coCartesian if and only if the morphism m_f is an isomorphism for each non-degenerate 1-simplex $f \in \mathcal{I}$.

3.1.2 Description of the functor $G^{\mathcal{I}}$

We are now in a position to describe the functor $G^{\mathcal{I}}$ constructed in [21, Proposition 5.1]. Given an object $m \in \lim \mathcal{D}$, $G^{\mathcal{I}}(m)$ gives us an $\mathcal{I}$-diagram in $\mathrm{Mod}_{\mathcal{C}}(R)$.

For each 0-simplex a in $\mathcal{I}$, the functor $\mathcal{R}(a) \otimes_R (-) : \mathrm{Mod}_{\mathcal{C}}(R) \to \mathrm{Mod}_{\mathcal{C}}(\mathcal{R}(a))$ has a right adjoint, denoted as G_a, which is the forgetful functor. With this notation, we have that $G^{\mathcal{I}}(m)(a) \simeq G_a(m_a)$.

For every 1-simplex $f : a \to b$ in $\mathcal{I}$, $G^{\mathcal{I}}(m)(f)$ is given by the composite

$$G_a(m_a) \xrightarrow{G_a(\eta_{m_a})} G_b(\mathcal{R}(f)^*m_a) \xrightarrow{G_b(m_f)} G_b(m_b),$$

where η is the unit of the tensor-forgetful adjunction

$$\mathrm{Mod}_{\mathcal{C}}(\mathcal{R}(a)) \underset{forget}{\overset{\mathcal{R}(f)^*}{\rightleftarrows}} \mathrm{Mod}_{\mathcal{C}}(\mathcal{R}(b)).$$

Unpacking this formula, and ignoring the forgetful right adjoints in the notation, we get that $G^{\mathcal{I}}(m)(f)$ as a map of R-modules is the composite

$$m_a \simeq \mathcal{R}(a) \otimes_{\mathcal{R}(a)} m_a \xrightarrow{\mathcal{R}(f)\otimes \mathbb{1}_{m_a}} R(b) \otimes_{R(a)} m_a \simeq \mathcal{R}(f)^*m_a \xrightarrow{m_f} m_b.$$

For a 2-simplex $\tau \in \mathcal{I}$ as above, we have that $G^{\mathcal{I}}(m)(\tau)$ is given by the 2-cell $G_c(m_\tau)$.

3.2 Construction of invertible $K(1)$-local spectra

Our primary objective in this section is to construct potential elements in the Picard group at height one. The verification of their invertibility will be carried out in Chapter 4.

For the purposes of this section, all spectra are $K(1)$-local at p. Furthermore, for a ring spectrum R, $\mathrm{Mod}(R)$ denotes the ∞-category of R-modules in $Sp_{K(1)}$.

We have a finite resolution of the $K(1)$-local sphere $L_{K(1)}S^0$

$$L_{K(1)}S^0 \longrightarrow K \underset{\mathbb{1}_K}{\overset{\psi_\gamma}{\rightrightarrows}} K\ .$$

Here, γ is a topological generator of $\mathbb{Z}_p^\times$, K denotes the p-completed topological complex K-theory, $\mathbb{1}_K$ denotes the identity map and ψ_γ denotes the Adams operation. Applying the functor $\mathrm{Mod} : \mathrm{CAlg}(Sp_{K(1)}) \to \mathcal{P}r^L$ we get

$$\mathfrak{A} : \mathrm{Mod}(L_{K(1)}S^0) \simeq Sp_{K(1)} \to \lim \left(\mathrm{Mod}(K) \underset{\mathbb{1}_K^*}{\overset{\psi_\gamma^*}{\rightrightarrows}} \mathrm{Mod}(K) \right). \tag{3.6}$$

For this particular example, $\mathcal{I} = \Delta_{\mathrm{inj},\leq 1}$ is our indexing category. The simplicial set $\mathcal{I}$ comprises of two 0-simplices and two non-degenerate 1-simplices. We denote this diagram as

$$0 \underset{g}{\overset{f}{\rightrightarrows}} 1.$$

Following the notation in Section 3.1.1, an object of the limit category on the right-hand side of 3.6 is a tuple (m_0, m_1, m_f, m_g), where $m_0, m_1 \in \mathrm{Mod}(K)$ and $m_f : \psi_\gamma^*(m_0) \to m_1$, $m_g : m_0 \to m_1$ are isomorphisms of K-modules. Utilizing the isomorphism m_g, the datum (m_0, m_1, m_f, m_g) amounts to a tuple of the form (M, ϕ), where M is a K-module, and $\phi : \psi_\gamma^*(M) \to M$ is an isomorphism of K-modules. Utilizing the formulae presented in Section 3.1.2, the right adjoint maps the object (M, ϕ) to the limit of the $\mathcal{I}$-diagram

$$\lim \left(M \simeq K\hat{\otimes}M \underset{\mathbb{1}_M}{\overset{\phi\circ(\psi_\gamma\hat{\otimes}\mathbb{1}_M)}{\rightrightarrows}} K\hat{\otimes}M \simeq M \right)$$

The limit of this diagram is given by taking the fiber of the map $M \xrightarrow{\phi\circ(\psi_\gamma\hat{\otimes}\mathbb{1}_M)-\mathbb{1}_M} M$. Since we are primarily interested in generating invertible elements, we can restrict the right adjoint to the Picard group of the limit category. In this case, we have the following corollary:

Corollary 3.2. *The elements in* $\mathrm{Pic}(Sp_{K(1)})$ *are of the form* $\Sigma^i X_\lambda$ *for* $i \in \{0,1\}$, $\lambda \in \mathbb{Z}_p^\times$ *and* X_λ *is the fiber in*

$$X_\lambda \to K \xrightarrow{\psi_\gamma - \lambda} K. \tag{3.7}$$

Proof. Firstly, we note that $\mathrm{Pic}(K) \simeq \mathbb{Z}/2$ is generated by ΣK. Therefore, the invertible objects in the limit are of the form $(\Sigma^i K, \phi)$ where i is either 0 or 1 and ϕ is K-module automorphism of ΣK. We conclude this proof by noting that the K-module automorphisms of K are simply given by multiplication of a unit in $\pi_0(K) \simeq \mathbb{Z}_p$. □

3.3 Construction of invertible $Q(l)$-modules

For the puposes of this section, fix an odd prime p, and all the spectra under consideration are $K(2)$-local at p. Furthermore, for a ring spectrum R, $\mathrm{Mod}(R)$ denotes the ∞-category of R-modules in $Sp_{K(2)}$. By l, we denote a topological generator of $\mathbb{Z}_p^\times$. The goal of this subsection is to prove Theorem 3.3 using the formula in Section 3.1.

First we recall from (2.31), that the spectrum $Q(l)$ is a limit of the diagram

$$Q(l) \coloneqq \lim \Big(TMF \rightrightarrows TMF_0(l) \times TMF \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} TMF_0(l) \Big) . \tag{3.8}$$

The face maps $d_i : TMF \to TMF_0(l) \times TMF$ are defined in (2.32), and the face maps $d_i : TMF_0(l) \times TMF \to TMF_0(l)$ are given in (2.34).

Applying the functor $\mathrm{Mod} : \mathrm{CAlg}(Sp_{K(2)}) \to \mathcal{P}r^L$, we get a functor

$$F : \mathrm{Mod}(Q(l)) \to \lim \Big(\mathrm{Mod}(TMF) \rightrightarrows \mathrm{Mod}(TMF_0(l)) \times \mathrm{Mod}(TMF) \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \mathrm{Mod}(TMF_0(l)) \Big) . \tag{3.9}$$

We denote by $\mathcal{L}$ the limit category which is the target of F in (3.9).

For this example, $\mathcal{I} = \Delta_{\mathrm{inj}, \leq 2}$ is the indexing category. The simplicial set $\mathcal{I}$ consists of three 0-simplices, eight non-degenerate 1-simplices and six non-degenerate 2-simplices. Equivalently it can thought of as the truncated semi cosimplicial diagram $0 \rightrightarrows 1 \substack{\to\\[-1ex]\to\\[-1ex]\to} 2$.

Following the notation in Section 3.1.1, an object of the limit category which is the target of the functor F in 3.6 is a tuple of modules (m_0, m_1, m_2) together with some compatibility data (as outlined in Section 3.1.1), where $m_0 \in \mathrm{Mod}(TMF)$, $m_1 \in \mathrm{Mod}(TMF_0(l)) \times \mathrm{Mod}(TMF)$ and $m_2 \in \mathrm{Mod}(TMF_0(l))$. As a part of the compatibility data, we have isomorphisms $d_1^* m_0 \simeq m_1$ and $d_2^* m_1 \simeq m_2$. Similar to the height one case in Section 3.2, using these isomorphisms, it suffices to consider tuples of the form $(M_0, \phi, \varepsilon_0, \varepsilon_1, \eta_1, \eta_2, \eta_3)$. Here $M_0 \in \mathrm{Mod}(TMF)$, $\phi : d_0^* M_0 \to d_1^* M_0$ is an isomorphism of $TMF \times TMF_0(l)$-modules, $\varepsilon_i : d_i^* d_1^* M_0 \to d_2^* d_1^* M_0$ are isomorphisms of $TMF_0(l)$-modules and η_i are homtopies that correspond to the three cosimplicial identities. We now describe the three homotopies η_i.

The cosimplicial identity $d_0 \circ d_1 = d_2 \circ d_0$ gives us a homotopy η_1 which fills the diagram in Figure 3.1.

$$\begin{tikzcd}
d_0^* d_1^* M_0 \simeq d_2^* d_0^* M_0 \arrow[rr, bend left, "\varepsilon_0"] \arrow[rr, bend right, "d_2^*\phi"'] \arrow[rr, phantom, "\Downarrow \eta_1"] & & d_2^* d_1^* M_0
\end{tikzcd}$$

Figure 3.1: The two simplex η_1

The cosimplicial identity $d_1 \circ d_1 = d_2 \circ d_1$ gives us a homotopy η_2 which fills the diagram in Figure 3.2.

$$\begin{tikzcd}
d_1^* d_1^* M_0 \simeq d_2^* d_1^* M_0 \arrow[rr, bend left, "\mathbb{1}_{d_2^* d_1^* M_0}"] \arrow[rr, bend right, "\varepsilon_1"'] \arrow[rr, phantom, "\Downarrow \eta_2"] & & d_2^* d_1^* M_0
\end{tikzcd}$$

Figure 3.2: The two simplex η_2

The cosimplicial identity $d_0 \circ d_0 = d_1 \circ d_0$ gives us a homotopy η_3 which fills the diagram in Figure 3.3.

$$
\begin{tikzcd}
d_1^*d_0^*M_0 \simeq d_0^*d_0^*M_0 \arrow[r, "d_0^*\phi"] \arrow[d, "d_1^*\phi"'] \arrow[dr, phantom, "\Downarrow \eta_3"] & d_0^*d_1^*M_0 \arrow[d, "\varepsilon_0"] \\
d_1^*d_1^*M_0 \arrow[r, "\varepsilon_1"'] & d_2^*d_1^*M_0
\end{tikzcd}
$$

Figure 3.3: The two simplex η_3

Theorem 3.3. *With $(M_0, \phi, \varepsilon_0, \varepsilon_1, \eta_1, \eta_2, \eta_3)$ as above we have the following:*

(1) $\phi : d_0^*M_0 \to d_1^*M_0$ satisfies the cocycle condition i.e the there exists a homotopy that fills the diagram in Figure 3.4.

$$
\begin{tikzcd}
d_0^*d_0^*M_0 \arrow[r, "d_0^*\phi"] \arrow[dr, "d_1^*\phi"'] \arrow[dr, phantom, "\Downarrow", shift left=4] & d_0^*d_1^*M_0 \simeq d_2^*d_0^*M_0 \arrow[d, "d_2^*\phi"] \\
 & d_1^*d_1^*M_0 \simeq d_2^*d_1^*M_0
\end{tikzcd}
$$

Figure 3.4: Cocycle condition

(2) The right adjoint G is given as the limit of the diagram

$$
\lim \left(\begin{tikzcd}
M_0 \arrow[r, shift left, "\phi\circ(d_0\hat{\otimes}\mathbb{1}_{M_0})"] \arrow[r, shift right, "d_1\hat{\otimes}\mathbb{1}_{M_0}"'] & d_1^*M_0 \arrow[r, shift left=2, "d_2^*(\phi)\circ(d_0\hat{\otimes}\mathbb{1}_{d_1^*M_0})"] \arrow[r, "d_1\hat{\otimes}\mathbb{1}_{d_1^*M_0}" description] \arrow[r, shift right=2, "d_2\hat{\otimes}\mathbb{1}_{d_1^*M_0}"'] & d_2^*d_1^*M_0
\end{tikzcd} \right)
$$

Proof. (1) follows immediately by putting the three homotopies η_i together in a diagram.

(2) Using the description of the right adjoint in Section 3.1.2, we get that the right adjoint G is given as the limit of the following diagram:

$$
\lim \left(\begin{tikzcd}
M_0 \arrow[r, shift left, "\phi\circ(d_0\hat{\otimes}\mathbb{1}_{M_0})"] \arrow[r, shift right, "d_1\hat{\otimes}\mathbb{1}_{M_0}"'] & d_1^*M_0 \arrow[r, shift left=2, "\varepsilon_0\circ(d_0\hat{\otimes}\mathbb{1}_{d_1^*M_0})"] \arrow[r, "\varepsilon_1\circ(d_1\hat{\otimes}\mathbb{1}_{d_1^*M_0})" description] \arrow[r, shift right=2, "d_2\hat{\otimes}\mathbb{1}_{d_1^*M_0}"'] & d_2^*d_1^*M_0
\end{tikzcd} \right)
$$

Using the homotopy η_1, we get that ε_0 is homotopic to $d_2^*\phi$. Further, using η_2 we get that ε_1 is homotopic to $\mathbb{1}_{d_2^*d_1^*M_0}$. This concludes the proof. $\square$

Since we are primarily interested in generating invertible elements, we can restrict the adjoint G to the Picard group of the limit category $\mathcal{L}$. In this case, we have the following corollary.

Corollary 3.4. *The elements in* $\mathrm{Pic}(Q(l))$ *are of the form* $G(n,b,a)$*, for* $a \in \pi_0(TMF)^\times$, $b \in \pi_0(TMF_0(l))^\times$ *satisfying* $\psi_d(b) \cdot b = a$ *and* $G(n,b,a)$ *is the limit*

$$G(n,b,a) = S^n \otimes \lim \left(\begin{tikzcd} TMF \arrow[r, shift left, "(b,a)\cdot d_0"] \arrow[r, shift right, "d_1"'] & TMF_0(l) \times TMF \arrow[r, shift left=2, "b\cdot d_0"] \arrow[r, "d_1" description] \arrow[r, shift right=2, "d_2"'] & TMF_0(l) \end{tikzcd} \right)$$

Proof. By [1], we note that $\mathrm{Pic}(TMF)$ is generated by ΣTMF. Therefore, the invertible objects in the limit are of the form $(\Sigma^i TMF, \phi)$ where i is an integer and ϕ is $TMF_0(l) \times TMF$-module automorphism of $\Sigma^i TMF_0(l) \times \Sigma^i TMF$. We conclude this proof by noting that, $TMF_0(l) \times TMF$-module automorphisms of $TMF_0(l) \times TMF$ are simply given by multiplication by the pair (b,a),where $b \in \pi_0(TMF_0(l))^\times$ and $a \in \pi_0(TMF)^\times$. The fact that ϕ satisfies the cocycle conditions gives us the relation $\psi_d(b) \cdot b = a$. $\square$

Remark 3.5. We caution that the converse of Corollary 3.4 may not be true. In particular, for specific values of n, a and b, the element $G(n,b,a)$ may not be an invertible $Q(l)$-module.

Chapter 4

Detection of invertible elements

In this chapter, we prove Theorem 4.11, a detection theorem for invertible $Q(l)$-modules. In Section 4.1, we recall Theorem 4.8 and Theorem 4.9 which were originally proved in [20]. In Section 4.2, we prove Theorem 4.10 and Theorem 4.11 analogously to the height one case.

The strategy we employ can be summarized as follows. In Theorem 4.10, we show that the map $f : Q(l) \to E_2$ admits descent in the sense of Definition 4.3 below. Using Theorem 4.10, we deduce a detection theorem for invertible $Q(l)$-modules in Theorem 4.11. Before delving into the height 2 case, we will also sketch the story at height one to illustrate our strategy.

We begin with the notion of a $\otimes$-ideal in a symmetric monoidal stable ∞-category $\mathcal{C}$. This notion has been extensively examined by Balmer in [2]. Our particular interest in this concept stems from its connection to Theorem 4.6, wherein Mathew demonstrates a link between the notion of descent and $\otimes$-ideals.

Definition 4.1 ([30, Definition 2.16])**.** *Let I be a subcategory of $\mathcal{C}$. We say that I is a $\otimes$-ideal if whenever $X \in \mathcal{C}$ and $Y \in I$, the tensor product $X \otimes Y \in \mathcal{C}$ actually belongs to I.*

We then introduce the concept of a thick $\otimes$-ideal.

Definition 4.2 ([30, Definition 3.17])**.** *If $(\mathcal{C}, \otimes, \mathbb{1})$ is a symmetric monoidal stable ∞-category, we will say that a full subcategory $\mathcal{D} \subset \mathcal{C}$ is thick if $\mathcal{D}$ is closed under finite limits, finite colimits and under retracts. In particular, $\mathcal{D}$ is stable. We say $\mathcal{D}$ is a thick $\otimes$-ideal if in addition it is a $\otimes$-ideal.*

With these definitions in place, we proceed to define the crucial notion of *descendability.*

Definition 4.3 ([30, Definition 3.18])**.** *If $(\mathcal{C}, \otimes, \mathbb{1})$ is a symmetric monoidal stable ∞-category. Given $A \in CAlg(\mathcal{C})$, we will say that A admits descent or is descendable if the thick $\otimes$-ideal generated by A is all of $\mathcal{C}$. More generally, we will say that a morphism $A \to B$ in $CAlg(\mathcal{C})$ admits descent if B, considered as a commutative algebra object in $\mathrm{Mod}_{\mathcal{C}}(A)$, admits descent in the above sense.*

Example 4.4. Here are a few well known descendability results in homotopy theory:

1. If $A \to B$ is a Galois extension of ring spectra (in the sense of [35]) with a finite Galois group, then it satsifies descent (see [14]).

2. The unit map $L_{K(n)}S^0 \to E_n$ admits $K(n)$-local descent (see [34]).

Remark 4.5. In [30, Proposition 3.20.], an alternative characterization of descent is established, and we will apply it in the proof of Theorem 4.10. Specifically, a commutative algebra $A \in \mathrm{CAlg}(\mathcal{C})$ admits descent if and only if the unit object $\mathbb{1} \in \mathcal{C}$ can be obtained as a retract of a finite colimit of a diagram in $\mathcal{C}$ consisting of objects, each of which admits the structure of an A-module.

We now state the crucial result about descendable objects, which will play a crucial role in establishing Theorem 4.10.

Theorem 4.6 ([30, Proposition 3.22])**.** *Let $R \to S$ be a descendable morphism in $CAlg(\mathcal{C})$. The the natural functor* $\mathrm{Mod}(R)$ *to the limit*

$$\mathrm{Mod}(R) \to \lim_{\Delta} \left\{ \mathrm{Mod}(S) \rightrightarrows \mathrm{Mod}(S \otimes_R S) \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \mathrm{Mod}(S \otimes_R S \otimes_R S) \substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow} \cdots \right\}$$

is an equivalence.

Informally, Theorem 4.6 tells us that we can construct any object in $\mathrm{Mod}(R)$ by gluing together S-modules.

4.1 Detection at height one

In this section, we will be working in the symmetric monoidal ∞-category of $K(1)$-local spectra, denoted as $(Sp_{K(1)}, \hat{\otimes}, L_{K(1)}S^0)$. When we have two $K(1)$-local spectra, X and Y, their tensor product in this symmetric monoidal category $X \hat{\otimes} Y := L_{K(1)}(X \otimes Y)$. It is worth noting that the smash product of two $K(1)$-local spectra might not be $K(1)$-local itself, which is why the subsequent $K(1)$-localisation step is crucial.

Further, by K we denote the $p-$completed topological complex K-theory. Note that K is already $K(1)$-local. The results in this section are well known, and and we include them here solely to demonstrate how they align with the framework of this thesis. We begin by recalling a descendability result for the $K(1)$-local sphere $L_{K(1)}S^0$.

Theorem 4.7 ([30, Proposition 10.10.])**.** *The map $L_{K(1)}S^0 \to K$ admits descent and*

$$Sp_{K(1)} \simeq \lim \left(\mathrm{Mod}(K) \rightrightarrows \mathrm{Mod}(K \hat{\otimes} K) \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \cdots \right),$$

where $\mathrm{Mod}(K)$ *denotes K-modules in the category of $K(1)$-local spectra,* $\mathrm{Mod}(K \hat{\otimes} K)$ *denotes $K \hat{\otimes} K$-modules in the category of $K(1)$-local spectra and so on.*

Below we state a detection result which was first proven in [20]. This is a very well known result. We are including it here because the proof for Theorem 4.11 will follow a similar pattern, making this proof an illustrative example of our method.

Theorem 4.8 ([20, Theorem 1.3.])**.** *If X is a $K(1)$-local spectrum, then the following are equivalent.*

1. *X is invertible in $(Sp_{K(1)}, \hat{\otimes}, L_{K(1)}S^0)$.*
2. *$K \hat{\otimes} X \simeq \Sigma^k K$ for some integer k.*
3. *$K(1) \hat{\otimes} X \simeq \Sigma^k K(1)$ for some integer k.*

Proof. This proof is not novel; in fact, it is essentially a restatement of the proof presented in [20]. We first show $1 \implies 3$. Let X be an invertible element. Then there is a Y such that $X \hat{\otimes} Y \simeq L_{K(1)}S^0$. Smashing with $K(1)$, we get

$$\begin{aligned} K(1) &\simeq K(1)\hat{\otimes}X\hat{\otimes}Y \\ K(1) &\simeq (K(1)\hat{\otimes}X)\hat{\otimes}_{K(1)}(K(1)\hat{\otimes}Y) \end{aligned} \tag{4.1}$$

Since $K(1)$ is a field we get that $K(1)\hat{\otimes}X \simeq \Sigma^k K(1)$ for some integer k.
We now show $2 \implies 1$. Applying $\mathfrak{pic}$ to the functor in Theorem 4.7, we get that X is invertible if and only if $X\hat{\otimes}K \in \mathrm{Pic}(K)$. However $\mathrm{Pic}(K)$ is generated by suspensions of K. Therefore, $X\hat{\otimes}K \simeq \Sigma^k K$ for some k.
$2 \implies 3$ is clear. To finish the proof we need to show $3 \implies 2$. This requires a version of Nakayama lemma which is stated in Theorem C.3. By Lemma C.1, $X\hat{\otimes}K$ is a complete K-module, which when reduced by the maximal ideal gives $\Sigma^k K(1)$. Applying Theorem C.3 to the module $M := X\hat{\otimes}K$, we get that $X\hat{\otimes}K \simeq \Sigma^k K$ for some k. □

We end this section with a proof of Theorem 4.9, which is a rephrasing of [20, Proposition 2.1].

Theorem 4.9. *The functor $\mathfrak{A}|_{\mathfrak{pic}}$ in (1.4) is an equivalence.*

Proof. We simply need to verify the invertibility of the elements X_λ defined in Corollary 3.2. To do this, we establish that $K(1)_*(X_\lambda) \simeq K(1)_*$. The element X_λ fits into a fiber sequence:

$$X_\lambda \to K \xrightarrow{\psi_\gamma - \lambda} K,$$

where γ is a topological generator of $\mathbb{Z}_p^\times$. By [11], we find that $K(1)_*(K) \simeq \mathrm{map}^c(\mathbb{Z}_p^\times, \mathbb{F}_p)[v_1^\pm]$. Under this identification, the action of ψ_γ is given by the formula

$$\psi_\gamma(f)(x) = f(\gamma x),$$

where $f : \mathbb{Z}_p^\times \to \mathbb{F}_p$ is a continuous function. Utilizing the definition of a continuous map from the profinite group $\mathbb{Z}_p^\times$, we can establish that $\psi_\gamma - \lambda$ is a surjective map from $\mathrm{map}^c(\mathbb{Z}_p^\times, \mathbb{F}_p)$ to itself, with its kernel being a one-dimensional $\mathbb{F}_p$-vector space (see Appendix B). Using the long exact sequence

$$K(1)_*(X_\lambda) \to (K(1))_*K \xrightarrow{\psi_\gamma - \lambda} (K(1))_*K,$$

we conclude that $(K(1))_*(X_\lambda) \simeq K(1)_*$. Therefore, by Theorem 4.8, we get that X_λ is invertible.

□

4.2 Detection at height two

In this section, we will be working in the symmetric monoidal ∞-category of $K(2)$-local spectra, denoted as $(Sp_{K(2)}, \hat{\otimes}, L_{K(2)}S^0)$. Similar to the height one case, given two $K(2)$-local spectra, X and Y, their tensor product in this symmetric monoidal category $X\hat{\otimes}Y := L_{K(2)}(X \otimes Y)$.

By [7], we have a map $f : Q(l) \to E_2$ by identifying $Q(l)$ as the homotopy fixed-points of E_2 with respect to the group Γ defined in [7]. In Theorem 4.10 we show that it satisfies descent.

Theorem 4.10. *The map $f : Q(l) \to E_2$ admits descent in the sense of [30, Definition 3.18] and*

$$\mathrm{Mod}(Q(l)) \simeq \lim \Big(\mathrm{Mod}(E_2) \rightrightarrows \mathrm{Mod}(E_2\hat{\otimes}_{Q(l)}E_2) \substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow} \mathrm{Mod}(E_2\hat{\otimes}_{Q(l)}E_2\hat{\otimes}_{Q(l)}E_2) \substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow} \cdots \Big),$$

where $\mathrm{Mod}(Q(l))$ *denotes* $Q(l)$*-modules in the category of* $K(2)$*-local spectra,* $\mathrm{Mod}(E_2)$ *denotes* E_2*-modules in the category of* $K(2)$*-local spectra,* $\mathrm{Mod}(E_2\hat{\otimes}_{Q(l)}E_2)$ *denotes* $E_2\hat{\otimes}_{Q(l)}E_2$*-modules in the category of* $K(2)$*-local spectra and so on.*

Proof. We will first show that that TMF is in the thick subcategory generated by E_2-modules in $\mathrm{Mod}(Q(l))$. According to [7, Section 5.5], we can express $K(2)$-local TMF as a finite product, $TMF \simeq \Pi_i E_2^{hK_i}$, where the K_i are finite subgroups of the Morava stabilizer group $\mathbb{G}_2$. By [14], we have that $E_2^{hK_i} \to E_2$, satisfies descent. Using Remark 4.5, we conclude that $E_2^{hK_i}$ is in the thick subcategory generated by E_2-modules in $\mathrm{Mod}(E_2^{hK_i})$. Forgetting the $E_2^{hK_i}$-module structure along the map $Q(l) \to E_2^{hK_i}$, it follows that $E_2^{hK_i}$ is in the thick subcategory generated by E_2-modules in $\mathrm{Mod}(Q(l))$. Given that TMF is a finite product of these $E_2^{hK_i}$, we get that TMF is in the thick subcategory generated by E_2-modules in $\mathrm{Mod}(Q(l))$.

Note that, every E_2-module N can be thought of as $Q(l)$-module via the map $Q(l) \to E_2$. Since N is a retract of $N\hat{\otimes}_{Q(l)}E_2$, every E_2-module is in the thick $\otimes$-ideal generated by E_2 in $\mathrm{Mod}(Q(l))$. Consequently, TMF is in the thick $\otimes$-ideal generated by E_2 in $\mathrm{Mod}(Q(l))$. Similarly we can show that $TMF_0(l)$ is contained in the thick $\otimes$-ideal generated by E_2 in $\mathrm{Mod}(Q(l))$. Since $Q(l)$ is a finite limit involving TMF and $TMF_0(l)$, the thick $\otimes$-ideal generated by E_2 contains $Q(l)$. This show that $Q(l) \to E_2$ satisfies descent and concludes the proof. □

Theorem 4.11. *If* $X \in \mathrm{Mod}(Q(l))$*, then the following are equivalent.*

1. *X is an invertible $Q(l)$-module.*
2. *$X\hat{\otimes}_{Q(l)}E_2 \simeq \Sigma^k E_2$ for some integer k.*
3. *$X\hat{\otimes}_{Q(l)}K(2) \simeq \Sigma^k K(2)$ for some integer k.*

Proof. We first show $1 \implies 3$. Let X be an invertible object. The map $Q(l) \to K(2)$, induces a map of Picard groups $\mathrm{Pic}(Q(l)) \to \mathrm{Pic}(K(2))$. Therefore, $X\hat{\otimes}_{Q(l)}K(2)$ is an invertible $K(2)$-module. Since $K(2)$ is a field we get that $K(2)\hat{\otimes}_{Q(l)}X \simeq \Sigma^k K(2)$ for some integer k.
We now show $2 \implies 1$. Applying $\mathfrak{pic}$ to the functor in Theorem 4.10, we get that X is invertible if and only if $X\hat{\otimes}_{Q(l)}E_2 \in \mathrm{Pic}(E_2)$. However $\mathrm{Pic}(E_2)$ is generated by suspensions of E_2. Therefore, $X\hat{\otimes}_{Q(l)}E_2 \simeq \Sigma^k E_2$ for some integer k.
$2 \implies 3$ is clear. To finish the proof we need to show $3 \implies 2$. This requires a version of Nakayama lemma which is stated in Theorem C.3. By Lemma C.1, $X\hat{\otimes}_{Q(l)}E_2$ is a complete E_2-module, which when reduced by the maximal ideal gives $\Sigma^k K(2)$. Applying Theorem C.3 to the module $M := X\hat{\otimes}_{Q(l)}E_2$, we get that $X\hat{\otimes}_{Q(l)}E_2 \simeq \Sigma^k E_2$ for some k. □

Even though we have this detection theorem at height two, the criterion is hard to verify and therefore we are unable to prove an analogous statement to Theorem 4.9.

Chapter 5

Spectral sequence for the Picard group of limit

In this chapter we work in the case where p is 3 and l is 2. In particular we will work with the spectrum $Q(2)$.

$$Q(2) := \lim \left(TMF \rightrightarrows TMF_0(2) \times TMF \substack{\rightarrow\\\rightarrow\\\rightarrow} TMF_0(2) \right). \tag{5.1}$$

Setting $l = 2$ in (3.9), we have a functor

$$F : \mathrm{Mod}(Q(2)) \to \lim \Big(\mathrm{Mod}(TMF) \rightrightarrows \mathrm{Mod}(TMF_0(2)) \times \mathrm{Mod}(TMF) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathrm{Mod}(TMF_0(2)) \Big), \tag{5.2}$$

and we denote by $\mathcal{L}$ the limit category, which is the target category of the functor F in (5.2).

Applying $\mathfrak{pic}$, we get a functor

$$F|_{\mathfrak{pic}} : \mathfrak{pic}(Q(2)) \to \mathfrak{pic}(\mathcal{L}). \tag{5.3}$$

Using the results in [31], our objective is to compute the Picard group of the limit $\mathcal{L}$, denoted $\mathrm{Pic}(\mathcal{L})$, through a Bousfield-Kan Spectral Sequence, as laid out in Theorem 5.1.

In Section 5.1, we describe the E_1-page of the Bousfiled-Kan spectral sequence. In Section 5.2, we compute the relevant terms on the E_2-page of the spectral sequence, allowing us to deduce Theorem 5.1. As a corollary, we deduce Corollary 5.5, which improves further on Corollary 3.4. Next, we discuss the group structure of $\mathrm{Pic}(\mathcal{L})$ in Section 5.3 and we prove Lemma 5.7. Finally, in Section 5.4, we prove Theorem 5.8, which not only relates invertible $Q(2)$-modules to the invertible $K(2)$-local spoectra constrcucted in [16] but also allows us to produce non-trivial invertible $Q(2)$-modules using the invertible $K(2)$-local spectra.

5.1 First page of the spectral sequence

In this section, we will describe the E_1-page of the Bousfield-Kan spectral sequence to compute $\mathrm{Pic}(\mathcal{L})$, the Picard group of the limit $\mathcal{L}$ in (5.2). By Theorem 2.14, we get that

$$\mathfrak{pic}(L) \simeq \lim \left(\mathfrak{pic}(TMF) \rightrightarrows \mathfrak{pic}(TMF_0(2)) \times \mathfrak{pic}(TMF) \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathfrak{pic}(TMF_0(2)) \right). \tag{5.4}$$

The Picard group $\mathrm{Pic}(\mathcal{L})$ is given by the 0th homotopy group of $\mathfrak{pic}(\mathcal{L})$, $\pi_0(\mathfrak{pic}(\mathcal{L}))$. The group $\pi_0(\mathfrak{pic}(\mathcal{L}))$ is computable via a Bousfield-Kan spectral sequence with E_1-page given by

$$E_1^{s,t} = \begin{cases} \pi_t(\mathfrak{pic}(TMF)), & \text{for } s = 0 \\ \pi_t(\mathfrak{pic}(TMF_0(2))) \oplus \pi_t(\mathfrak{pic}(TMF)), & \text{for } s = 1 \\ \pi_t(\mathfrak{pic}(TMF_0(2))), & \text{for } s = 2 \\ 0 & \text{for } s > 2. \end{cases} \tag{5.5}$$

Due to the vanishing line above $s = 2$, the spectral sequence degenerates at page 3 and $E_3^{s,t} = E_\infty^{s,t}$.

5.1.1 *Computation of the relevant E_1-terms*

Our main focus lies in calculating $\pi_0(\mathfrak{pic}(\mathcal{L}))$, allowing us to narrow down computations to the vicinity of $t - s = 0$ on the E_1-page. In particular, we need to compute the terms with $t - s = 0$ i.e the terms $\pi_0(\mathfrak{pic}(TMF))$, $\pi_1(\mathfrak{pic}(TMF_0(2))) \oplus \pi_1(\mathfrak{pic}(TMF))$ and $\pi_2(\mathfrak{pic}(TMF_0(2)))$. We also need to consider the differentials, along with their sources and targets, that have the above three terms as either the source or the target.

All the relevant E_1-terms follow easily once we know the zeroth and first homotopy groups of the Picard spectra $\mathfrak{pic}(TMF)$ and $\mathfrak{pic}(TMF_0(2))$. By [31, Example 2.2.2] and using equations (2.38) and (2.39), we get that

$$\pi_2(\mathfrak{pic}(TMF_0(2))) \simeq \pi_1(TMF_0(2)) \simeq 0 \tag{5.6}$$

$$\pi_1(\mathfrak{pic}(TMF)) \simeq \pi_0(TMF)^\times \simeq \mathbb{Z}_3[[j]]^\times \tag{5.7}$$

$$\pi_1(\mathfrak{pic}(TMF_0(2))) \simeq \pi_0(TMF_0(2))^\times \simeq \mathbb{Z}_3[[x]]^\times \tag{5.8}$$

Further using [18, Theorem 4.1], we get that $\mathrm{Pic}(TMF)$ is a cyclic group of order 72 generated by ΣTMF and $\mathrm{Pic}(TMF_0(2))$ is a cyclic group of order 8 generated by $\Sigma TMF_0(2)$.

5.1.2 *Description of the d_1-differentials*

The d_1-differentials in the spectral sequence are an alternating sum of the face maps (see [17, Chapter VIII]). However, since we are only interested in the 0th homotopy group, we only need to consider three possible differentials. The relevant portion of the spectral sequence is shown below in Figure 5.1.

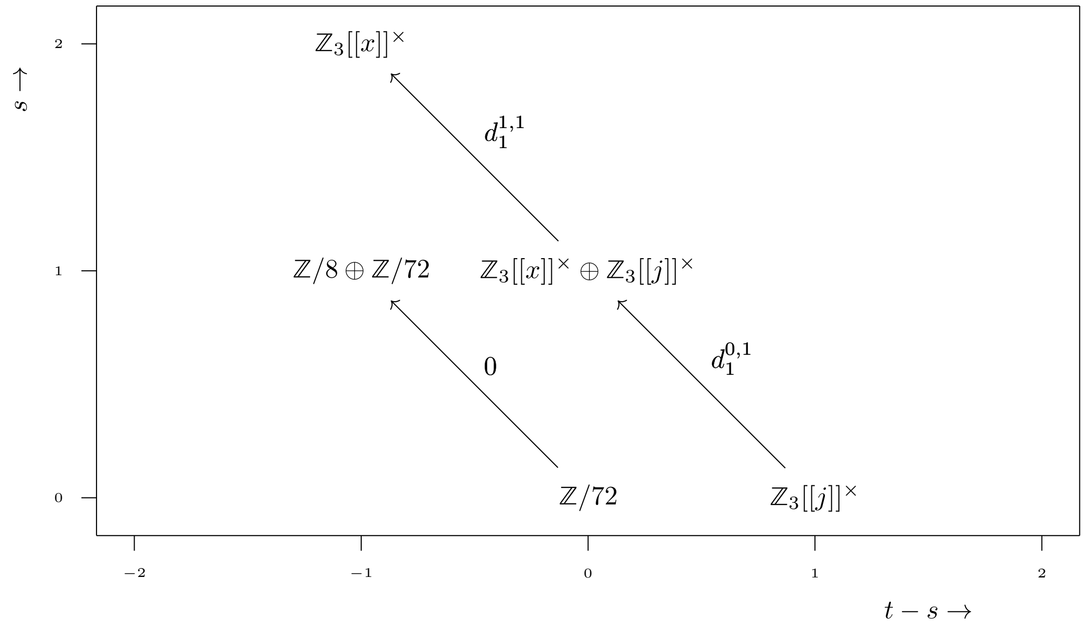

Figure 5.1: BKSS for Pic(Q(2))

First, consider the differential $d_1^{0,0} : E_1^{0,0} \to E_1^{1,0}$, where $E_1^{0,0} = \mathrm{Pic}(TMF)$ and $E_1^{1,0} = \mathrm{Pic}(TMF_0(2)) \oplus \mathrm{Pic}(TMF)$. The differential $d_1^{0,0}$ is a difference of two maps: $\left(\phi_q^* \oplus \psi_{[2]}^*\right)|_{\mathrm{Pic}(TMF)}$ and $\left(\phi_f^* \oplus \mathbb{1}_{TMF}^*\right)|_{\mathrm{Pic}(TMF)}$. Here, $\left(\phi_q^* \oplus \psi_{[2]}^*\right)|_{\mathrm{Pic}(TMF)}$ and $\left(\phi_f^* \oplus \mathbb{1}_{TMF}^*\right)|_{\mathrm{Pic}(TMF)}$ are obtained by applying the functor Mod followed by restricting to the Picard group $\mathrm{Pic}(TMF)$ for the maps $\phi_q \oplus \psi_{[2]}$ and $\phi_f \oplus \mathbb{1}_{TMF}$ respectively (as defined in (2.22),(2.24),(2.27) and (2.23)).

We claim that the differential $d_1^{0,0}$ is trivial. To prove this it suffices to show that the generator ΣTMF of $\mathrm{Pic}(TMF)$ maps to zero under the map $d_1^{0,0}$. To see this, we note that the maps ϕ_f^*, ϕ_q^*, $\psi_{[2]}^*$, and $\mathbb{1}_{TMF}^*$ are all symmetric monoidal functors and therefore map the unit object to the unit object. Consequently, we can deduce that $\left(\phi_q^* \oplus \psi_{[2]}^*\right)(\Sigma TMF) \simeq \left(\phi_f^* \oplus \mathbb{1}_{TMF}^*\right)(\Sigma TMF) \simeq \Sigma TMF_0(2) \oplus \Sigma TMF$. As a result, $d_1^{0,0}$ evaluates to zero.

Next, consider the differential $d_1^{1,1} : E_1^{1,1} \to E_1^{2,1}$. By Corollary 2.16, this differential is given by the map

$$\begin{aligned} d_1^{1,1} : \pi_0(TMF_0(2))^\times \oplus \pi_0(TMF))^\times &\to \pi_0 TMF_0(2))^\times \\ (b, a) &\to \frac{\psi_d\,(b) \cdot b}{a} \end{aligned} \tag{5.9}$$

Finally, we have the differential $d_1^{0,1} : E_1^{0,1} \to E_1^{1,1}$. By Corollary 2.16, this differential is given by the map

$$\begin{aligned} d_1^{0,1} : \pi_0(TMF)^\times &\to \pi_0(TMF_0(2))^\times \oplus \pi_0(TMF)^\times \\ \alpha &\to \left(\frac{\psi_d\,(\alpha)}{\alpha}, 1\right) \end{aligned} \tag{5.10}$$

The Atkin-Lehner map restricted to π_0 acts as an involution. Specifically, the map $\psi_d|_{\pi_0} : \pi_0(TMF_0(2)) \to \pi_0(TMF_0(2))$ satisfies $\psi_d|_{\pi_0}^2 = \mathbb{1}_{\pi_0(TMF_0(2))}$. Consequently, through simple algebraic manipulations, we find that $d_1^{1,1} \circ d_1^{0,1} = 0$.

Our main computational result about the Picard group of the limit category is the following.

Theorem 5.1. *The natural map* $\mathrm{Pic}(\mathcal{L}) \to \mathrm{Pic}(TMF)$ *is surjective and we get a short exact sequence*

$$0 \to E_2^{1,1} \to \mathrm{Pic}(\mathcal{L}) \to \mathrm{Pic}(TMF) \to 0, \tag{5.11}$$

where $\mathrm{Pic}(TMF) \simeq \mathbb{Z}/72$ *is generated by* ΣTMF. *The kernel* $E_2^{1,1}$ *is a quotient of the subgroup of* $\pi_0(TMF_0(2))^\times \times \pi_0(TMF)^\times$ *consisting of elements of the form* $\{(b, a) : \psi_d(b) \cdot b = a\}$, *where* ψ_d *is the Atkin-Lehner involution defined in* (2.23), $b \in \pi_0(TMF_0(2))^\times$ *and* $a \in 1 + 3\mathbb{Z}_3 \subset \mathbb{Z}_3^\times \subset \pi_0(TMF)^\times$. *Furthermore, as a group,* $E_2^{1,1} \simeq \mathbb{Z}_3^\times \oplus \left(\frac{\pi_0(TMF_0(2))^\times}{Im(\psi_d \cdot id)\pi_0(TMF)^\times}\right)$, *where* $Im\,(\psi_d \cdot id)$ *is the image of the map defined in* (5.27).

The primary challenge in establishing this theorem lies in the calculation of the $E_2^{1,1}$ term of the spectral sequence in (5.5), a computation we undertake in the following section.

5.2 Second page of the spectral sequence

In this section, we compute the $E_2^{1,1}$ term of the spectral sequence. On the zeroth homotopy group, the map ϕ_f induces an injection $\pi_0(TMF) \hookrightarrow \pi_0(TMF_0(2))$, enabling us to regard $\pi_0(TMF)$ as a subring of $\pi_0(TMF_0(2))$.

$E_2^{1,1}$ is given by the homology group $\frac{ker\left(d_1^{1,1}\right)}{Im\left(d_1^{0,1}\right)}$, where $ker\left(d_1^{1,1}\right)$ is the kernel of the map in (5.9) and $Im\left(d_1^{0,1}\right)$ is the image of the map in (5.10). Using the definition in (5.9), we get that

$$ker\left(d_1^{1,1}\right) = \{(b, a) : \psi_d\,(b) \cdot b = a\}, \tag{5.12}$$

where ψ_d is the Atkin-Lehner involution defined in (2.23), $b \in \pi_0(TMF_0(2))^\times$ and $a \in \pi_0(TMF)^\times$.

The key to calculating the $E_2^{1,1}$ term is the following result, which restricts the possible values of a.

Theorem 5.2. *With notation as above, if* $(b, a) \in ker\left(d_1^{1,1}\right)$, *then* $a \in \pi_0(TMF)^\times = \mathbb{Z}_3[[j]]^\times$ *is a constant power series.*

Proof. Since (b, a) is an element of the kernel we have

$$a = \psi_d(b) \cdot b$$

Applying ψ_d to this equation and noting that ψ_d acts as an involution on $\pi_0(TMF_0(2))^\times$, we get $\psi_d\,(a) = a$. Suppose $a \in \pi_0(TMF)^\times$ has the form $a = c + \sum_{i=1}^{\infty} a_i j^i \in \mathbb{Z}_3[[j]]$ with the constant term c. Since $\psi_d\,(a) = a$, we obtain the following equation:

$$\sum_{i=1}^{\infty} a_i (j^i - \psi_d\,(j)^i) = 0. \tag{5.13}$$

If all a_i are divisible by 3, we can divide the equation by 3 and repeat this process until there exists an a_i that is not divisible by 3. If this process continues indefinitely, it implies that all a_i are divisible by increasingly larger powers of 3, leading them to converge to 0 in $\mathbb{Z}_3$. In this case the theorem holds trivially.

Thus, without loss of generality, we can assume that 3 does not divide all a_i. Reducing equation (5.13) modulo 3 and using formulae (3) and (4) in Lemma 2.35, we get

$$\sum_{i=1}^{\infty} a_i \left((-1)^i \left(\frac{x^3}{1-x} \right)^i - \left(\frac{x^3}{(1-x)^2} \right)^i \right) \equiv 0 \pmod 3. \tag{5.14}$$

Now, our goal is to show that $a_i \equiv 0 \pmod 3$ for all i, which leads to a contradiction since we assumed that 3 does not divide all the a_i. We will prove this by induction on the 3-adic valuation of i.
Base case of the Induction:
Claim: If $a = c + \sum_{i=1}^{\infty} a_i j^i \in \mathbb{F}_3[[j]]$ satisfies $\psi_d(a) = a$, then $a_i = 0$ whenever $3 \nmid i$.

Proof of Claim. For any $\phi \in \mathbb{F}_3[[x]]$, we have that $\phi^3 \in \mathbb{F}_3[[x^3]]$. Therefore, $j^i, \psi_d(j)^i \in \mathbb{F}_3[[x^3]]$ whenever $3 \mid i$. However, the terms where $3 \nmid i$ give rise to terms not in $\mathbb{F}_3[[x^3]]$. We first show that $a_1 = 0$. For this we will consider the equation (5.14) modulo x^6. By Lemma 2.35 and further simplification using the binomial theorem we get

$$j^i - \psi_d(j)^i \equiv -2x^3 - x^5 \mod (3, x^6) \text{ for } i = 1, \tag{5.15}$$

$$j^i - \psi_d(j)^i \equiv 0 \mod (3, x^6) \text{ for } i > 2. \tag{5.16}$$

Therefore, equation (5.14) reduces to $-2a_1x^3 - a_1x^5 \equiv 0 \mod x^6$. Thus we conclude that $a_1 = 0$. Now suppose we have shown that $a_i = 0$ for all $i \le n$ which are not divisible by 3. Then, the next term after a_n in the set $\{a_i : 3 \nmid i\}$ is either a_{n+1} or a_{n+2}. We assume here that the next term is a_{n+1}. The proof when the next term is a_{n+2} is similar. To show $a_{n+1} = 0$ we consider two cases:
Case 1: $n+1$ is even
In this case, we consider the equation (5.14) modulo x^{3n+5}. By Lemma 2.35 and further simplification using the binomial theorem we get

$$j^i - \psi_d(j)^i \equiv -(n+1)x^{3n+4} \mod (3, x^{3n+5}) \text{ for } i = n+1, \tag{5.17}$$

$$j^i - \psi_d(j)^i \equiv 0 \mod (3, x^{3n+5}) \text{ for } i > n+1. \tag{5.18}$$

Therefore, equation (5.14) reduces to $-(n+1)a_{n+1}x^{3n+4} + \phi = 0$ modulo x^{3n+5}, where $\phi \in \mathbb{F}_3[[x^3]]$. Thus we conclude that $a_{n+1} = 0$.
Case 2: $n+1$ is odd
In this case, we consider the equation (5.14) modulo x^{3n+6}. By Lemma 2.35 and further simplification using the binomial theorem we get

$$j^i - \psi_d(j)^i \equiv -2x^{3n+3} - (n+1)^2x^{3n+5} \mod (3, x^{3n+6}) \text{ for } i = n+1, \tag{5.19}$$

$$j^i - \psi_d(j)^i \equiv 0 \mod (3, x^{3n+6}) \text{ for } i > n+1. \tag{5.20}$$

Therefore, equation (5.14) reduces to $-(n+1)^2a_{n+1}x^{3n+5} + \zeta = 0$ modulo x^{3n+6}, where $\zeta \in \mathbb{F}_3[[x^3]]$. Thus we conclude that $a_{n+1} = 0$. ☐

Inductive Step:
Assume that $a_i = 0$ whenever $\nu_3(i) < k$. Each a_i that satisfies $\nu_3(i) \ge k$ can be written as $a_{3^k \cdot j}$ and thus

equation (5.14) simplifies to

$$\sum_{j=1}^{\infty} a_{3^k.j}\left((-1)^j\left(\frac{x^3}{1-x}\right)^{3^k.j}-\left(\frac{x^3}{(1-x)^2}\right)^{3^k.j}\right)=0. \tag{5.21}$$

Setting $b_j := a_{3^k.j}$ and $y := x^{3^k}$, equation (5.21) can be rewritten as

$$\sum_{j=1}^{\infty} b_j\left((-1)^j\left(\frac{y^3}{1-y}\right)^{j}-\left(\frac{y^3}{(1-y)^2}\right)^{j}\right)=0. \tag{5.22}$$

Comparing equation (5.22) with (5.14) and using the same argument as in the base case, we conclude that $b_i = 0$ whenever $3 \nmid i$. Hence, we conclude that $a_i = 0$ when $\nu_3(i) = k$. This concludes the inductive step. Therefore, $a_i = 0$ for all i. This is a contradiction, since we assumed that 3 does not divide all the a_i. So a has to be a constant power series. $\square$

As a consequence of Theorem 5.2, we have the following corollary which restricts the possible values of a even further.

Corollary 5.3. *If $(b, a) \in ker\left(d_1^{1,1}\right)$, then a is a square in $\mathbb{Z}_3[[x]]^\times$. In particular, $a \in 1 + 3\mathbb{Z}_3 \subset \mathbb{Z}_3^\times$.*

Proof. From Remark 2.36 it follows that $\psi_d(b) \equiv b \pmod{(3, x)}$. Since $a = \psi_d(b)\cdot b$, we have that a is a square by Lemma 2.37. $\square$

Moving forward, we will compute the $E_2^{1,1}$ term. We have an inclusion

$$\begin{aligned} i : \mathbb{Z}_3^\times &\hookrightarrow E_2^{1,1} \\ u &\to [(u, u^2)] \end{aligned} \tag{5.23}$$

Next, we construct a section s for the map i. By Corollary 5.3, a is a square, and we can now consider the following map

$$\begin{aligned} \widetilde{s} : ker\left(d_1^{1,1}\right) &\to \mathbb{Z}_3^\times \\ (b, a) &\to \left(\frac{b}{|b|}\sqrt{a}\right), \end{aligned} \tag{5.24}$$

with $\sqrt{a}$ and $|b|$ as defined in definitions 2.38 and 2.40. Under $\widetilde{s}$ the group $Im\left(d_1^{0,1}\right)$ maps to the identity in $\mathbb{Z}_3^\times$. Therefore the map $\widetilde{s}$ factors through the quotient group $E_2^{1,1}$.

$$\begin{tikzcd} ker\left(d_1^{1,1}\right) \arrow[rr, "\widetilde{s}"] \arrow[dr, "q"'] & & \mathbb{Z}_3^\times \\ & E_2^{1,1} \arrow[ur, dashed, "s"'] & \end{tikzcd}$$

Clearly $s \circ i$ is the identity. Therefore, we get

$$E_2^{1,1} \simeq (E_2^{1,1}/\mathbb{Z}_3^\times) \oplus \mathbb{Z}_3^\times. \tag{5.25}$$

We will end this section with a description of the quotient group $E_2^{1,1}/\mathbb{Z}_3^{\times}$. However, prior to that, we require some notation. We have a group homomorphism denoted

$$\begin{aligned} \frac{\psi_d}{id} : \pi_0(TMF_0(2))^{\times} &\to \pi_0(TMF_0(2))^{\times} \\ \alpha &\to \frac{\psi_d\left(\alpha\right)}{\alpha} \end{aligned} \tag{5.26}$$

Let $ker\left(\frac{\psi_d}{id}\right)$ and $Im\left(\frac{\psi_d}{id}\right)$ denote the kernel and the image of this map repectively.

We have a group homomorphism denoted

$$\begin{aligned} \psi_d \cdot id : \pi_0(TMF_0(2))^{\times} &\to \pi_0(TMF_0(2))^{\times} \\ \alpha &\to \psi_d\left(\alpha\right) \cdot \alpha \end{aligned} \tag{5.27}$$

Let $Im\left(\psi_d \cdot id\right)$ denote the image of this map.

Theorem 5.4. *With notation as above we have the following:*

(1) $ker\left(d_1^{1,1}\right)/\mathbb{Z}_3^{\times} \simeq Im\left(\frac{\psi_d}{id}\right)$

(2) $E_2^{1,1}/\mathbb{Z}_3^{\times} \simeq \frac{\pi_0(TMF_0(2))^{\times}}{ker\left(\frac{\psi_d}{id}\right)\pi_0(TMF)^{\times}}$

(3) $E_2^{1,1}/\mathbb{Z}_3^{\times} \simeq \frac{\pi_0(TMF_0(2))^{\times}}{Im(\psi_d \cdot id)\pi_0(TMF)^{\times}}$

(4) $E_2^{1,1} \simeq \mathbb{Z}_3^{\times} \oplus \left(\frac{\pi_0(TMF_0(2))^{\times}}{Im(\psi_d \cdot id)\pi_0(TMF)^{\times}}\right)$

Proof. To establish statement (1), we begin with an observation: If $x \in \pi_0(TMF_0(2))^{\times}$ is a square satisfying $\psi_d\left(x\right) \cdot x = 1$, then we have $x \in Im\left(\frac{\psi_d}{id}\right)$. This observation follows by noting that

$$x = \sqrt{x \cdot \psi_d\left(x\right)} \cdot \sqrt{\frac{x}{\psi_d\left(x\right)}} = \frac{\sqrt{x}}{\psi_d\left(\sqrt{x}\right)}. \tag{5.28}$$

If $(b, a) \in ker\left(d_1^{1,1}\right)$, then we have $\psi_d\left(\frac{|b|}{\sqrt{a}}\right) \cdot \frac{|b|}{\sqrt{a}} = 1$. Applying the observation above to $\frac{|b|}{\sqrt{a}}$, we conclude that $\frac{|b|}{\sqrt{a}} \in Im\left(\frac{\psi_d}{id}\right)$.

Consequently, we introduce the map

$$\begin{aligned} f : ker\left(d_1^{1,1}\right) &\to Im\left(\frac{\psi_d}{id}\right) \\ (b, a) &\to \frac{|b|}{\sqrt{a}} \end{aligned} \tag{5.29}$$

We assert that the map f is surjective, and its kernel is exactly the group $i(\mathbb{Z}_3^{\times})$. For any $\alpha \in Im\left(\frac{\psi_d}{id}\right)$, we have that $(\alpha, 1) \in ker\left(d_1^{1,1}\right)$, thereby proving the surjectivity of f. By Theorem 5.2, we have that a is a constant. Elementary algebraic computations show that the kernel of f is precisely given by the subgroup $\mathbb{Z}_3^{\times}$. This concludes the proof of statement (1).

Now we proceed to establish statement (2). By the group isomorphism theorem, we have an isomorphism $g : Im\left(\frac{\psi_d}{id}\right) \simeq \pi_0(TMF_0(2))^\times / ker\left(\frac{\psi_d}{id}\right)$. Therefore, it suffices to calculate the image of the group $Im\left(d_1^{0,1}\right)$ under $g \circ f$. Using the definition of the differential in (5.10), we deduce that the group $Im\left(d_1^{0,1}\right)$ is generated by elements of the form $(\frac{\psi_d(\alpha)}{\alpha}, 1)$, where $\alpha \in \pi_0(TMF)^\times$. Under f these elements map to $\frac{\psi_d(\alpha)}{\alpha}$. Using the fact that ψ_d is an involution, we get $\frac{\psi_d}{id}\left(\sqrt{\frac{\alpha}{\psi_d(\alpha)}}\right) = \frac{\psi_d(\alpha)}{\alpha}$. Therefore, under $g \circ f$ the group $Im\left(d_1^{0,1}\right)$ maps to the group generated by elements of the form $[\sqrt{\frac{\alpha}{\psi_d(\alpha)}}]$. Since $\psi_d(\alpha) \cdot \alpha \in ker\left(\frac{\psi_d}{id}\right)$, the two equivalence classes $[\alpha]$ and $[\sqrt{\frac{\alpha}{\psi_d(\alpha)}}]$ coincide. Thus, we have that the image of the group $Im\left(d_1^{0,1}\right)$ under $g \circ f$ is given by $\pi_0(TMF)^\times$. This concludes the proof of statement (2).

For (3), it suffices to show that $\pi_0(TMF)^\times Im\,(\psi_d \cdot id)$ is isomorphic to $\pi_0(TMF)^\times ker\left(\frac{\psi_d}{id}\right)$. Since, $Im\,(\psi_d \cdot id) \subset ker\left(\frac{\psi_d}{id}\right)$, we conclude that $\pi_0(TMF)^\times Im\,(\psi_d \cdot id) \subset \pi_0(TMF)^\times ker\left(\frac{\psi_d}{id}\right)$. We now show they are isomorphic by proving that $ker\left(\frac{\psi_d}{id}\right) \subset \pi_0(TMF)^\times Im\,(\psi_d \cdot id)$. In particular, we claim that if $a \in ker\left(\frac{\psi_d}{id}\right)$, then $|a| \in Im\,(\psi_d \cdot id)$. This follows by noting that

$$|a| = \sqrt{\frac{|a|}{\psi_d\,(|a|)}} \cdot \sqrt{|a| \cdot \psi_d\,(|a|)} = \sqrt{|a|} \cdot \psi_d\left(\sqrt{|a|}\right). \tag{5.30}$$

Therefore, $|a| \in Im\,(\psi_d \cdot id)$. This concludes the proof of statement (3).

Statement (4) follows by combining statement (3) and (5.25). $\square$

Using the multiplicative basis of $\pi_0(TMF_0(2))^\times$, Theorem 5.4 allows us to write a algorithm for computing the $E_2^{1,1}$ term. However, the algorithm is still computationally intensive and we do not pursue it in this thesis.

Before we present the proof of Theorem 5.1, we depict the relevant part of the E_2-page in Figure 5.2.

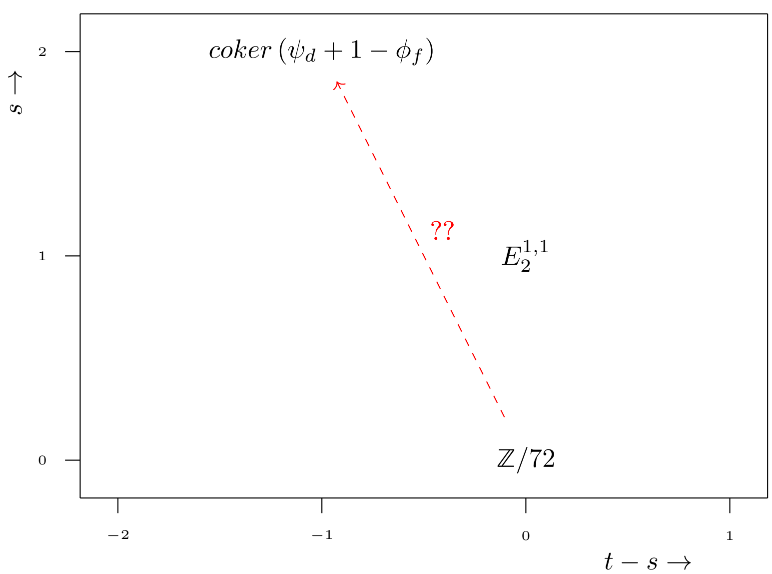

Figure 5.2: Relevant part of E_2-page of BKSS for Pic(Q(2))

The dashed red line represents the only possible differential, while *coker* $\left(d_1^{1,1}\right)$ represents the cokernel of the map $d_1^{1,1}$ introduced in (5.9). In the subsequent proof, we show that the dashed differential is trivial. This completes the proof of Theorem 5.1.

Proof of Theorem 5.1. We show that there can be no non-trivial differential from $\mathbb{Z}/72$. To prove this, it suffices to show that the map $g : \mathrm{Pic}(\mathcal{L}) \to \mathrm{Pic}(TMF)$ is a surjection. Applying Pic to the unit map $\eta : L_{K(2)}S^0 \to Q(2)$, gives rise to a map of Picard groups $\mathrm{Pic}(\eta) : \mathrm{Pic}_2 \to \mathrm{Pic}(Q(2))$, where Pic_2 is the Picard group of the $K(2)$-local category of spectra. Composing with the map $\mathrm{Pic}(F)$ (with F as defined in (5.2)), we get $\mathrm{Pic}(F)\circ\mathrm{Pic}(\eta) : \mathrm{Pic}_2 \to \mathrm{Pic}(\mathcal{L})$. Further, composing with g, we get $g\circ\mathrm{Pic}(F)\circ\mathrm{Pic}(\eta) : \mathrm{Pic}_2 \to \mathrm{Pic}(TMF)$, which is clearly a surjection. Therefore, g is surjective.

By Theorem 5.4, we conclude that the group $E_2^{1,1}$ is isomorphic to $\mathbb{Z}_3^\times \oplus \left(\frac{\pi_0(TMF_0(2))^\times}{\pi_0(TMF)^\times Im(\psi_d \cdot id)}\right)$. This concludes our proof.

□

As an immediate consequence, we obtain an enhancement of Corollary 3.4. In particular, we are able to restrict the choices for the element a below from $\pi_0(TMF)^\times$ to $1 + 3\mathbb{Z}_3 \subset \mathbb{Z}_3^\times$.

Corollary 5.5. *The elements in* $\mathrm{Pic}(Q(2))$ *are of the form* $G(n, b, a)$*, for* n *an integer* $\pmod{72}$*,* $b \in \pi_0(TMF_0(2))^\times$*,* $a \in 1 + 3\mathbb{Z}_3 \subset \mathbb{Z}_3^\times \subset \pi_0(TMF)^\times$ *such that* $\psi_d(b) \cdot b = a$ *and* $G(n, b, a)$ *is the limit*

$$G(n,b,a) = S^n \otimes \lim \left(\begin{tikzcd} TMF \arrow[r, shift left, "(b,a)\cdot d_0"] \arrow[r, shift right, "d_1"'] & TMF_0(2) \times TMF \arrow[r, shift left=2, "b\cdot d_0"] \arrow[r, "d_1" description] \arrow[r, shift right=2, "d_2"'] & TMF_0(2) \end{tikzcd} \right)$$

5.3 *Group structure of* $\mathrm{Pic}(\mathcal{L})$

In this section, we will describe the group structure of $\mathrm{Pic}(\mathcal{L})$ and use the group structure to prove Lemma 5.7. By Theorem 5.1, an element in $\mathrm{Pic}(\mathcal{L})$ can be represented by an equivalence class $[(n, b, a)]$. Here, n is an integer, $b \in \pi_0(TMF_0(2))^\times$, and $a \in 1 + 3\mathbb{Z}_3 \subset \mathbb{Z}_3^\times \subset \pi_0(TMF)^\times$ such that $\psi_d(b) \cdot b = a$. The product of two elements, $[(n_1, b_1, a_1)]$ and $[(n_2, b_2, a_2)]$, is simply $[(n_1 + n_2, b_1 \cdot b_2, a_1 \cdot a_2)]$.

Let us explore when two equivalence classes describe the same element in $\mathrm{Pic}(\mathcal{L})$.

Corollary 5.6. *With notation as above,*

1. *The two representatives,* $[(n, b, a)]$ *and* $[(n, b', a')]$*, represent the same element if and only if both conditions hold:* $a = a'$ *and* $b = b' \cdot \frac{\psi_d(\alpha)}{\alpha}$ *for some* $\alpha \in \pi_0(TMF_0(2))^\times$.
2. *The two representatives*$[(72, 1, 1)]$ *and* $[(0, \frac{\Delta^3}{\psi_d(\Delta^3)}, \frac{\Delta^3}{\psi_{[2]}(\Delta^3)})]$ *represent the same element in* $\mathrm{Pic}(\mathcal{L})$.

Proof. According to Theorem 5.1, the two representatives, $[(n, b, a)]$ and $[(n, b', a')]$, represent the same element if and only if both conditions hold: $a = a'$ and $b = b' \cdot \frac{\psi_d(\alpha)}{\alpha}$ for some $\alpha \in \pi_0(TMF_0(2))^\times$. Further, by Theorem 5.1, the element in $\mathrm{Pic}(\mathcal{L})$ represented by $[(72, 1, 1)]$ also has a representative of the form $[0, b, a]$. To find such a representative we start with the map $\Delta^3 : \Sigma^{72}TMF \to TMF$ given by multiplication by Δ^3.

Now we find (b, a) that make the diagram below commute

$$\begin{tikzcd}
\Sigma^{72}TMF_0(2) \times \Sigma^{72}TMF \arrow[r, "\Sigma^{72}\mathbb{1}_{TMF_0(2)} \times \Sigma^{72}\mathbb{1}_{TMF}"] \arrow[d, "\psi_d(\Delta^3)" ', "\times \psi_{[2]}(\Delta^3)"] & \Sigma^{72}TMF_0(2) \times \Sigma^{72}TMF \arrow[d, "\Delta^3" ', "\times \Delta^3"] \\
TMF_0(2) \times TMF \arrow[r, "(b,a)"] & TMF_0(2) \times TMF
\end{tikzcd}$$

Simple algebraic computation yields that $(b, a) = (\frac{\Delta^3}{\psi_d(\Delta^3)}, \frac{\Delta^3}{\psi_{[2]}(\Delta^3)})$. Therefore, we get that $[(72, 1, 1)]$ and $[(0, \frac{\Delta^3}{\psi_d(\Delta^3)}, \frac{\Delta^3}{\psi_{[2]}(\Delta^3)})]$ represent the same element in $\mathrm{Pic}(\mathcal{L})$. $\square$

Now we are ready to prove Lemma 5.7 which states that the extension in Theorem 5.1 is non-trivial.

Lemma 5.7. *The extension in Theorem 5.1 is non-trivial.*

Proof. We will prove this by contradiction. Suppose the extension was trivial. Then there is element of order 72 represented by the tuple $[(1, b, a)]$. Raising $[(1, b, a)]$ to the 72 power, we get $[(72, b^{72}, a^{72})]$.

By Corollary 5.6, we get that the classes $[(72, b^{72}, a^{72})]$ and $[(0, \frac{\Delta^3}{\psi_d(\Delta^3)} b^{72}, \frac{\Delta^3}{\psi_{[2]}(\Delta^3)} a^{72})]$ are the same. Since this is the identity, we have

$$[(0, \frac{\Delta^3}{\psi_d(\Delta^3)} b^{72}, \frac{\Delta^3}{\psi_{[2]}(\Delta^3)} a^{72})] = [(0, 1, 1)].$$

Again, by Corollary 5.6, this is possible if and only if

$$\frac{\Delta^3}{\psi_{[2]}(\Delta^3)} a^{72} = 1 \tag{5.31}$$

$$\frac{\Delta^3}{\psi_d(\Delta^3)} b^{72} = \frac{\psi_d(\alpha)}{\alpha}, \tag{5.32}$$

where $\alpha \in \pi_0(TMF)^\times$. Since $\frac{\psi_{[2]}(\Delta^3)}{\Delta^3} = 2^{36}$, equation (5.31) simplifies to $a^2 = 2$. This is a contradiction, since 2 is not a square in $\mathbb{Z}_3^\times$. Therefore, the extension in Theorem 5.1 is non-trivial. $\square$

5.4 Relation to the invertible objects in the $K(2)$-local category of spectra

In this section, we will relate the invertible objects in the $K(2)$-local category of spectra to the invertible $Q(2)$-modules. We have a map $\mathrm{Pic}(\eta) : \mathrm{Pic}_2 \to \mathrm{Pic}(Q(2))$ which is induced by the unit map of $Q(2)$. We know that the algebraic part of Pic_2 is generated by $S^0, S^0\langle det\rangle$ and the exotic part is generated by P and Q [16, 23]. Using the construction of the spectra P and $S^0\langle det\rangle$ in [16] and [3] respectively, we calculate the images of various elements in Pic_2 in Theorem 5.8.

Theorem 5.8. *The spectra $S^0\langle det\rangle$ and P are detected in $\mathrm{Pic}(\mathcal{L})$ and therefore are also detected in $\mathrm{Pic}(Q(2))$. The image of $S^0\langle det\rangle$ in $\mathrm{Pic}(Q(2))$ is given by $G(0, 2, 4)$. The image of P is given by $G(48, \frac{\psi_d(\Delta^2)}{\Delta^2}, 2^{24})$, where ψ_d is the Atkin-Lehner involution.*

Proof. To compute the images of the generators of Pic_2 in $\mathrm{Pic}(Q(2))$, we first compute their images in $\mathrm{Pic}(\mathcal{L})$. Since the functor F in (5.2) is fully faithful, we can obtain the image in $\mathrm{Pic}(Q(2))$ by applying the right adjoint G.

We will first compute the image of $S^0\langle det\rangle$. By [3], we get that $K(2)$-locally $TMF \otimes S^0\langle det\rangle \simeq TMF$ and $TMF_0(2) \otimes S^0\langle det\rangle \simeq TMF_0(2)$. However, tensoring with $S^0\langle det\rangle$ twists the G_2-action on TMF and $TMF_0(2)$ by the determinant map defined in 2.48. Thus, it suffices to compute the determinant of $\psi_{[2]}$ and ψ_d. By [12, Proposition 2.19], the element $\psi_{[2]}$ is represented by the matrix $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$. Therefore, the determinant is equal to 4. Similarly, using [12, Proposition 2.19], the element ψ_d is represented by the matrix $\left(\begin{smallmatrix} 1+\omega^2 & 0 \\ 0 & 1+\omega^6 \end{smallmatrix}\right)$, where ω is a fixed chosen primitive a 8th root of unity in the Witt vectors $W = W(\mathbb{F}_{p^2})$. Therefore, the determinant is equal to 2. Thus, the image of $S^0\langle det\rangle$ in $\mathrm{Pic}(Q(2))$ is given by $G(0, 2, 4)$.

To calculate image of P, we will use results from [16, Theorem 5.5]. By [16, Theorem 5.5], we get that $P \otimes TMF \simeq \Sigma^{48}TMF$. Therefore, the image of P in $\mathrm{Pic}(\mathcal{L})$ is of the form $[(48, b, a)]$. Further, [16, Theorem 5.5] yields that the map $P \otimes TMF \simeq \Sigma^{48}TMF$ induces an isomorphism of the Morava modules $\mathrm{map}^c(G_2/G_{24}, (E_2)_*) \to \mathrm{map}^c(G_2/G_{24}, \Sigma^{48}(E_2)_*)$ given by multiplication by Δ^{-2}. Thus, we get a commutative diagram

$$\begin{tikzcd}
(E_2)_*(\Sigma^{48}TMF_0(2)) \times (E_2)_*(\Sigma^{48}TMF) \arrow[r, "\Sigma^{48}b \times \Sigma^{48}a"] \arrow[d, "\psi_d(\Delta^2)"', "\times \psi_{[2]}(\Delta^2)"] & (E_2)_*(\Sigma^{48}TMF_0(2)) \times (E_2)_*(\Sigma^{48}TMF) \arrow[d, "\Delta^2"', "\times \Delta^2"] \\
(E_2)_*(TMF_0(2)) \times (E_2)_*(TMF) \arrow[r, "(1,1)"] & (E_2)_*(TMF_0(2)) \times (E_2)_*(TMF)
\end{tikzcd}$$

Simple algebra yields that $(b, a) = (\frac{\psi_d(\Delta^2)}{\Delta^2}, 2^{24})$. Therefore, we get that the image of P is $G(48, \frac{\psi_d(\Delta^2)}{\Delta^2}, 2^{24})$. $\square$

An important consequence of this theorem is that, we have now produced non-trivial elements in $\mathrm{Pic}(Q(2))$.

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Appendix A

Descent for $K(1)$-local spectra

For the purposes of this section, fix an odd prime p and assume that all spectra are $K(1)$-local at p. Moreover, for a ring spectrum R, $\mathrm{Mod}(R)$ denotes the ∞-category of R-modules in the category of $K(1)$-local spectra $Sp_{K(1)}$. Let $\mathcal{I}$ denote the simplicial indexing category $\Delta_{\mathrm{inj},\leq 1}$, which consists of two 0-simplices and two non-degenerate 1-simplices.

The main result of this section is that the functor $\mathfrak{A}$ defined in (3.6) is not an equivalence.

Theorem A.1. *At an odd prime p, the functor*

$$\mathfrak{A} : Sp_{K(1)} \to \lim \left(\mathrm{Mod}(K) \overset{\psi_\gamma^*}{\underset{\mathbb{1}_K^*}{\rightrightarrows}} \mathrm{Mod}(K) \right). \tag{A.1}$$

is **NOT** *an equivalence.*

Remark A.2. In [9, proposition 3.9], the authors prove a similar result for the $K(1)$-local category of spectra. However, in their case, the authors replace the p-completed topological k-theory with the Adams L-summand and relate the category of $K(1)$-local spectra $Sp_{K(1)}$ with the limit category $\mathrm{Mod}(L)^{h\mathbb{Z}}$, where the $\mathbb{Z}$-action is given by the Adams operation. Specifically, they prove that $Sp_{K(1)}$ forms a strict subcategory, and its essential image is precisely the subcategory of L-modules equipped with a continuous action of $\mathbb{Z}_p$.

The similarity is further highlighted by noting that the limit category $\mathrm{Mod}(L)^{h\mathbb{Z}}$ is equivalent to the limit category $\lim\left(\mathrm{Mod}(L) \rightrightarrows \mathrm{Mod}(L)\right)$, where one arrow is the identity and the second map is the generator of the $\mathbb{Z}$-action.

We prove Theorem A.1 by showing that the right adjoint of $\mathfrak{A}$ is not conservative. However, before delving into this proof, we revisit some key results from Section 3.2.

Recall from Section 3.2 that an object of the limit category on the right-hand side of A.1 is a tuple of the form (M, ϕ), where M is a K-module, and $\phi : \psi_\gamma^*(M) \to M$ is an isomorphism of K-modules. Furthermore, the right adjoint maps the object (M, ϕ) to the limit of the $\mathcal{I}$-diagram

$$\lim \left(M \simeq K \hat{\otimes} M \overset{\phi \circ (\psi_\gamma \hat{\otimes} \mathbb{1}_M)}{\underset{\mathbb{1}_M}{\rightrightarrows}} K \hat{\otimes} M \simeq M \right)$$

The limit of this diagram is given by taking the fiber of the map $M \xrightarrow{\phi\circ(\psi_\gamma \hat{\otimes} \mathbb{1}_M) - \mathbb{1}_M} M$. Therefore, it suffices to produce a non-trivial module M and an isomorphism ϕ, such that the fiber of $M \xrightarrow{\phi\circ(\psi_\gamma \hat{\otimes} \mathbb{1}_M)-\mathbb{1}_M} M$ is trivial.

We are now in a position to prove Theorem A.1 and we give the proof below.

Proof of Theorem A.1. Using [21, Example 4.2, Lemma 4.7], we get a functor

$$\widetilde{\mathfrak{A}} : Sp^{\mathcal{I}}_{K(1)} \hookrightarrow \mathrm{Lax}\left(\mathrm{Mod}(K) \underset{1}{\overset{\psi^*_\gamma}{\rightrightarrows}} \mathrm{Mod}(K) \right), \tag{A.2}$$

where $Sp^{\mathcal{I}}_{K(1)}$ is the ∞-category of $\mathcal{I}$-diagrams in $Sp_{K(1)}$ and the target of the functor $\widetilde{\mathfrak{A}}$ is the lax limit (see [21, Definition 4.1]). Consider the $\mathcal{I}$-diagram

$$L_{K(1)}S^0 \oplus L_{K(1)}S^0 \rightrightarrows L_{K(1)}S^0 \oplus L_{K(1)}S^0,$$

where the two face maps are given by given by

$$d_0 = \tau \circ (p_1, \gamma \cdot p_2) \tag{A.3}$$

$$d_1 = id. \tag{A.4}$$

Here, τ is the swap or braid map which switches the coordinates and p_1, p_2 are the projections onto the first and second component respectively.

The image of this $\mathcal{I}$-diagram under $\widetilde{\mathfrak{A}}$ is $(K \oplus K, \phi)$, where the isomorphism $\phi : K \oplus K \to K \oplus K$ is given by $\tau \circ (p_1, \gamma \cdot p_2)$. In matrix form, ϕ is represented by $\left(\begin{smallmatrix} 0 & \gamma \\ 1 & 0 \end{smallmatrix}\right)$. We claim that the image of $(K \oplus K, \phi)$ under the right adjoint is trivial. In particular, we show that the fiber $K \oplus K \xrightarrow{\phi\circ\psi_\gamma - 1} K \oplus K$ is trivial by proving that the map $(\phi \circ \psi_\gamma - 1)$ induces an isomorphism on the homotopy groups.

Note that, for each $n \in \mathbb{Z}$, we have $\pi_{2n}(K \oplus K) = \mathbb{Z}_p \oplus \mathbb{Z}_p$. Furthermore, the map induced on homotopy groups by ψ_γ is given by the matrix $\left(\begin{smallmatrix} \gamma^n & 0 \\ 0 & \gamma^n \end{smallmatrix}\right)$. Therefore, the induced map on homotopy groups for $(\phi \circ \psi_\gamma - 1)$ is given by

$$\left(\begin{smallmatrix} 0 & \gamma \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} \gamma^n & 0 \\ 0 & \gamma^n \end{smallmatrix}\right) - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -1 & \gamma^{n+1} \\ \gamma^n & -1 \end{smallmatrix}\right)$$

This matrix is invertible if and only if the determinant modulo p is non-zero. The determinant of this matrix is $\gamma^{2n+1} - 1$. Since γ has order $p-1$ modulo p, the determinant is 0 modulo p if and only if $p - 1/2n+1$. However, this is not possible when the prime p is odd. Therefore, the matrix is invertible. This concludes the proof of Theorem A.1.

□

Appendix B

Continuous maps from $\mathbb{Z}_p^\times$

In this section, we explore the map $\psi_\gamma - \lambda : K \to K$ described in Section 3.2. Specifically, we prove that the map $\psi_\gamma - \lambda$ induces a surjection from 0th $K(1)$-homology of K, $K(1)_0(K)$ to itself. This result is well-known (see [27, Lecture 35]), and we provide a proof for the sake of completeness.

By [11], we get $K(1)_0(K) \simeq \mathrm{map}^c(\mathbb{Z}_p^\times, \mathbb{F}_p)$. Under this identification, the action of ψ_γ is given by the formula

$$\psi_\gamma(f)(x) = f(\gamma x),$$

where $f : \mathbb{Z}_p^\times \to \mathbb{F}_p$ is a continuous function.

Theorem B.1. *Let γ be a topological generator of $\mathbb{Z}_p^\times$ and $\lambda \in \mathbb{Z}_p^\times$. Then the map $\psi_\gamma - \lambda$ induces a surjection from* $\mathrm{map}^c(\mathbb{Z}_p^\times, \mathbb{F}_p)$ *to itself. Further, the kernel of this map is $\mathbb{F}_p$.*

Proof. Let f be an element in the kernel of $(\psi_\gamma - \lambda)$. Then $f(\gamma x) = \lambda f(x)$ for all $x \in \mathbb{Z}_p^\times$. Since γ is a topological generator, f is determined by its value at 1. Therefore, the kernel is $\mathbb{F}_p$.

Now we will show that $(\psi_\gamma - \lambda)$ is a surjection. Let $g : \mathbb{Z}_p^\times \to \mathbb{F}_p$ be any continuous map. Since g is continuous, it must factor through the quotient $\mathbb{Z}_p^\times/p^n(p-1)$ for some $n > 0$, , giving us a map $\hat{g} : \mathbb{Z}_p^\times/p^n(p-1) \to \mathbb{F}_p$. Composing this with the quotient map $q : \mathbb{Z}_p^\times/p^{n+1}(p-1) \to \mathbb{Z}_p^\times/p^n(p-1)$, yields a map $\bar{g} : \mathbb{Z}_p^\times/p^{n+1}(p-1) \to \mathbb{F}_p$.

$$\begin{tikzcd}
\mathbb{Z}_p^\times \arrow[rr, "g"] \arrow[dr] & & \mathbb{F}_p \\
& \mathbb{Z}_p^\times/p^n(p-1) \arrow[ur, "\hat{g}"'] & \\
\mathbb{Z}_p^\times/p^{n+1}(p-1) \arrow[ur, "q"'] \arrow[uurr, bend right, "\bar{g}"'] & &
\end{tikzcd}$$

Our strategy involves constructing a map $f : \mathbb{Z}_p^\times/p^{n+1}(p-1) \to \mathbb{F}_p$, such that $(\psi_\gamma - \lambda) f = \bar{g}$. Since γ is a topological generator of $\mathbb{Z}_p^\times$, we can represent all the elements in $\mathbb{Z}_p^\times/p^{n+1}(p-1)$ as γ^i for $0 \le i \le p^{n+1}(p-1) - 1$.

Consider the function f defined by the formula

$$f(\gamma^i) := \begin{cases} \sum_{j=0}^{i-1} \lambda^j \bar{g}(\gamma^{i-1-j}), & \text{for } 0 < i < p^{n+1}(p-1) \\ 0, & \text{for } i = 0 \end{cases}$$

We claim that this function f satisfies $(\psi_\gamma - \lambda) f = \bar{g}$. In particular, we need to verify that

$$(\psi_\gamma - \lambda)(f)(\gamma^i) = \bar{g}(\gamma^i) \text{ for } 0 \leq i < p^{n+1}(p-1). \tag{B.1}$$

Simple algebraic manipulation yields that equation B.1 is satisfied for $0 \leq i < p^{n+1}(p-1) - 1$. Therefore, it suffices to show that $(\psi_\gamma - \lambda)(f)(\gamma^{p^{n+1}(p-1)-1}) = \bar{g}(\gamma^{p^{n+1}(p-1)-1})$. Expanding we need to verify that,

$$f(1) - \lambda f(\gamma^{p^{n+1}(p-1)-1}) = \bar{g}(\gamma^{p^{n+1}(p-1)-1}). \tag{B.2}$$

Further expanding using the definition of f, it suffcies to show that

$$\sum_{j=0}^{p^{n+1}(p-1)-1} \lambda^j \bar{g}(\gamma^{p^{n+1}(p-1)-1-j}) = 0. \tag{B.3}$$

Since $\bar{g}$ was a lift of $\hat{g}$, we get $\bar{g}(\gamma p^n(p-1) + a) = \bar{g}(\gamma^a)$. Grouping terms, we get

$$\sum_{j=0}^{p^{n+1}(p-1)-1} \lambda^j \bar{g}(\gamma^{p^{n+1}(p-1)-1-j}) = \sum_{j=0}^{p^n(p-1)-1} \lambda^{p^{n+1}(p-1)-1-j} \bar{g}(\gamma^j) \left(\sum_{i=0}^{p-1} \lambda^{-ip^n(p-1)} \right). \tag{B.4}$$

Since $\lambda^{p-1} = 1$, we get $\left(\sum_{i=0}^{p-1} \lambda^{-ip^n(p-1)} \right) = 0$. This concludes the proof. $\square$

Appendix C

L-complete modules

Let E_n be the n-th Morava E-theory and let $K(n)$ denote the n-th Morava K-theory. For the purposes of this section, we assume all spectra are $K(n)$-local and and I_n denotes the maximal ideal $(v_0, v_1, \dots, v_{n-1})$ in E_{n_*} In this section, we will discuss two well-established results. In Lemma C.1, we demonstrate that $K(n)$-local E_n-modules have L-complete homotopy groups, following the approach outlined in [22].

The proof we provide for Lemma C.1 is due to Neil Strickland. A similar result is also established in [4, Corollary 3.14], although their proof is different. Finally, we will prove a version of the Nakayama lemma following closely the ideas in [22].

Lemma C.1. *If M is a $K(n)$-local E_n-module, then the homotopy groups $\pi_*(M)$ are L-complete in the sense of [22] with respect to the maximal ideal $I_n = (v_0, v_1, \dots, v_{n-1})$ in E_{n_*}.*

Proof. By [22, Proposition 7.10], we get that $K(n)$-localization is equivalent to taking limit over generalized Moore spectra S/I. Therefore, we have

$$\lim_I{}^1 \pi_{*+1}(M \wedge S/I) \hookrightarrow \pi_*(M) \twoheadrightarrow \lim_I \pi_*(M \wedge S/I).$$

The identity map of $E_n \wedge S/I$ satisfies $\mathfrak{m}^r.\mathbb{1}_{E_n \wedge S/I} = 0$ for $r \gg 0$, so $\mathfrak{m}^r.\pi_k(M \wedge S/I) = 0$ for $r \gg 0$, so $\pi_k(M \wedge S/I)$ is L-complete. By [22, Theorem A.6(g)], we get that both $\lim_I{}^1 \pi_{*+1}(M \wedge S/I)$ and $\lim_I \pi_{*+1}(M \wedge S/I)$ are L-complete. Moreover, extensions of L-complete modules are L-complete, by [22, Theorem A.6(e)]. It follows that $\pi_*(M)$ is L-complete. □

Remark C.2. By [4, Corollary 3.14], one, in fact, has that an E_n-module is $K(n)$-local if and only if its homotopy groups are L-complete.

Theorem C.3 ([22]). *Let M be an E_n-module. If $M \otimes_R R/I_n \simeq R/I_n$, then $M \simeq R$.*

Proof. We will closely follow the strategy of [22, Proposition 2.5]. We have that $\pi_*(M \otimes_{E_n} E_n/I_n) = E_{n_*}/I_n$. We also have a cofibre sequence

$$\Sigma^{|v_k|} E_n/I_k \to E_n/I_k \to E_n/I_{k+1}.$$

When $k+1 = n$, we have

$$\pi_*(M \otimes_{E_n} E_n/I_{n-1})/v_{n-1} \hookrightarrow \pi_*(M \otimes_{E_n} E_n/I_n),$$

and since the homotopy groups are concentrated only in even degrees we get, $\pi_{\mathrm{odd}}(M \otimes_{E_n} E_n/I_{n-1})/v_{n-1} = 0$. By Lemma C.1, we have that $\pi_*(M \otimes_{E_n} E_n/I_{n-1})$ is L-complete. Therefore, by [22, Proposition A.8], we get $\pi_{\mathrm{odd}}(M \otimes_{E_n} E_n/I_{n-1}) = 0$. A similar argument shows that v_{n-1} acts injectively on $\pi_{\mathrm{even}}(M \otimes_{E_n} E_n/I_{n-1})$ and therefore on $\pi_*(M \otimes_{E_n} E_n/I_{n-1})$. This implies $\pi_*(M \otimes_{E_n} E_n/I_{n-1})/(v_{n-1}) \simeq \pi_*(M \otimes_{E_n} E_n/I_n)$. By induction, we get that $\pi_*(M \otimes_{E_n} E_n)/(v_0, ..., v_{n-1}) \simeq \pi_*(M \otimes_{E_n} E_n/I_n)$. It follows from [22, Theorem A.9] that $\pi_*(M)$ is precisely E_{n_*}.

□

Appendix D

Code for computing invariant modular forms

```
# This is the code for checking psi(a)=a implies a is a constant
prec = 90  # Set the precision for the power series
a = []     # Initialize the list for the power series coefficients

# Fill the list 'a' with coefficients (all ones in this case)
for i in range(prec):
    a.append(1)
a

# Define a function to multiply two power series modulo 3
def mult(a: list, b: list):
    c = []
    for i in range(prec):
        m = 0
        for j in range(i + 1):
            m = m + (a[j] * b[i - j])
        c.append(m % 3)
    return c

# Loop over different powers
for i in range(1, 100):
    power = i
    b = a
    # Calculate b = a^(2*power-1)
    for i in range(2 * power - 1):
        b = mult(b, a)
    c = a
```

```python
# Calculate c = a^(power-1)
for i in range(power - 1):
    c = mult(c, a)
ans = []
# Check if power is even or odd to calculate the difference or sum
if power % 2 == 0:
    for i in range(prec):
        ans.append(b[i] - c[i])
else:
    for i in range(prec):
        ans.append(b[i] + c[i])

lead_power = 0
# Find the leading non-zero coefficient
for i in range(prec):
    if ans[i] % 3 != 0:
        lead_power = i
        break

output = ""
# Construct the output power series as a string
for i in range(prec):
    if ans[i] % 3 != 0:
        output = output + str(ans[i]) + "x^" + str(i - lead_power) + "+"

# Print the results for each power
print("The power series at n=" + str(power) + " is x^" + str(3 * power +
    lead_power) + "(" + output + ")")
```